Markov processes and martingales
2nd midterm test

10th May 2021

1. Let X be an exponential random variable with parameter 1, that is with density $f(x) = e^{-x}$ for $x \geq 0$. Define the random variables $Y_\lambda = e^{\lambda X}$ for any $\lambda \in \mathbb{R}$.
   (a) (3 points) Show that for any $\mu < 1$ fixed, the family of random variables $\{Y_\lambda, \lambda \leq \mu\}$ is uniformly integrable. Hint: one of the sufficient conditions for uniform integrability can be checked.
   (b) (3 points) Prove directly by definition that the class $\{Y_\lambda, \lambda < 1\}$ is not uniformly integrable.
   (c) (2 points) Argue that the almost sure limit $\lim_{\lambda \to 1} Y_\lambda = Y_1$ cannot be a limit in L^1.
2. (7 points) Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbf{E}(X_1) = -\infty$. This means that $\mathbf{E}(X_1^+) < \infty$ and $\mathbf{E}(X_1^-) = \infty$ where $X_1 = X_1^+ - X_1^-$ and $X_1^+, X_1^- \geq 0$. Let $S_n = X_1 + \cdots + X_n$. Prove that $S_n/n \to -\infty$ a.s. as $n \to \infty$. *Hint:* For any $M > 0$ let $X_i^M = \max(X_i, -M)$ and let $S_n^M = X_n^M + \cdots + X_n^M$. By the strong law of large numbers S_n^M/n converges and it can be used to upper bound S_n/n.