Exercises in Markov processes and martingales*

2020/21 spring semester

Conditional expectation

1. We roll two dices. X is the result of one of them and Z the sum of the results. Find $\mathbf{E}[Z|X]$ and $\mathbf{E}[X|Z]$.

2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where $\Omega = [0,1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbf{P}$ is the Lebesgue measure on it. Define the random variables $X(\omega) = 3\omega^2$ and $Y(\omega) = \mathbb{1}(\omega \in [1/2, 1]) - \mathbb{1}(\omega \in [0, 1/2))$ for any $\omega \in \Omega$. What is $\mathbf{E}(X|Y)$?

3. Let $\Omega = \{-1, 0, +1\}$, $\mathcal{F} = 2^\Omega$ and $\mu(\{-1\}) = \mu(\{0\}) = \mu(\{+1\}) = 1/3$. Consider also the sub-$\sigma$-algebras

$$\mathcal{G} = \{\emptyset, \{-1\}, \{0, +1\}, \Omega\}, \quad \mathcal{H} = \{\emptyset, \{-1, 0\}, \{+1\}, \Omega\}.$$

   Let $X : \Omega \to \mathbb{R}$ be the random variable $X(\omega) = \omega$. Compute $\mathbf{E}(\mathbf{E}(X|\mathcal{G})|\mathcal{H})$ and $\mathbf{E}(\mathbf{E}(X|\mathcal{H})|\mathcal{G})$.

4. **Homework 1.A. (22nd Feb)** Let X_j, $j = 1, 2, \dots$ be iid. random variables with common distribution $\mathbf{P}(X_j = -1) = \mathbf{P}(X_j = +1) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. Compute the conditional expectations $\mathbf{E}(X_1|S_n)$, $\mathbf{E}(S_n|X_1)$ and $\mathbf{E}(S_{n+m}^2|S_n)$.

5. Suppose that the random variables X, Y, Z are jointly defined on a probability space. Prove that

   (a) $\mathbf{E}(X) = \mathbf{E}(\mathbf{E}(X|Y))$,

   (b) $\mathbf{E}(Y|Z) = \mathbf{E}(\mathbf{E}(Y|X, Z)|Z)$.

6. Prove the following general version of Bayes's formula. Given the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let $G \in \mathcal{G}$ and $A \in \mathcal{F}$ with $\mathbf{P}(A) > 0$. Show that

$$\mathbf{P}(G|A) = \frac{\int_G \mathbf{P}(A|\mathcal{G})\,\mathrm{d}\mathbf{P}}{\int_\Omega \mathbf{P}(A|\mathcal{G})\,\mathrm{d}\mathbf{P}}.$$

7. Prove the conditional variance formula

$$\mathrm{Var}(X) = \mathbf{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbf{E}[\mathrm{X}|\mathrm{Y}])$$

   where $\mathrm{Var}(X|Y) = \mathbf{E}[X^2|Y] - (\mathbf{E}[X|Y])^2$.

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8. **Homework 1.B. (22nd Feb)** Let $X_1, X_2, \ldots$ be iid. random variables and N be a non-negative integer valued random variable which is independent of X_i, $i \geq 1$. Prove that

$$\mathrm{Var}\left(\sum_{i=1}^{N} X_i\right) = \mathbf{E}[N]\,\mathrm{Var}(X) + (\mathbf{E}[X])^2\mathrm{Var}(N).$$

9. Let X be a random variable. Assume that Y is another random variable for which $\mathbf{P}(Y = 0 \text{ or } Y = 1) = 1$. Prove that $Y \in \sigma(X)$ iff there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ Borel measurable function such that $Y = \varphi(X)$.

10. Assume that X, Y are jointly continuous random variables with joint density $f(x, y)$. Prove that

$$\mathbf{E}[Y|X] = \frac{\int_{\mathbb{R}} y f(X, y)\,\mathrm{d}y}{\int_{\mathbb{R}} f(X, y)\,\mathrm{d}y}.$$

11. Let $Y \in \sigma(\mathcal{G})$. Prove that

$$\mathbf{E}[X|\mathcal{G}] \geq Y \iff \forall A \in \mathcal{G}\ \mathbf{E}[X \cdot \mathbb{1}_A] \geq \mathbf{E}[Y \cdot \mathbb{1}_A].$$

12. **Homework 1.C. (22nd Feb)** Let X and Y be random variables on the same probability space. Prove that X and Y are independent iff for every $\varphi : \mathbb{R} \to \mathbb{R}$ bounded measurable functions we have

$$\mathbf{E}[\varphi(Y)|X] = \mathbf{E}[\varphi(Y)].$$

13. Let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$\mathbf{E}[X|Y] = Y \text{ and } \mathbf{E}[Y|X] = X.$$

Show that $\mathbf{P}(X = Y) = 1$.

Hint: If $X \neq Y$, then there is a $c \in \mathbb{Q}$ such that either $X \leq c$ and $Y > c$ or $X > c$ and $Y \leq c$.

Martingales

14. **Homework 2.A. (1st Mar)** Let X_t be the counting process of a Poisson point process with rate $\lambda = 1$. (See Durrett's book p. 139 for the definition.) Find $\mathbf{E}[X_1|X_2]$ and $\mathbf{E}[X_2|X_1]$.

15. **Homework 2.B. (1st Mar)** Let $S_n := X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are iid. with $X_1 \sim \mathrm{Exp}(1)$. Verify that

$$M_n := \frac{n!}{(1 + S_n)^{n+1}} \cdot e^{S_n}$$

is a martingale with respect to the natural filtration $\mathcal{F}_n$.

16. **Homework 2.C. (1st Mar)** For every $i = 1, \ldots, m$ let $\left\{M_n^{(i)}\right\}_{n=1}^{\infty}$ be a sequence of martingales with respect to $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Show that $M_n := \max_{1 \leq i \leq m} M_n^{(i)}$ is a submartingale with respect to $\{\mathcal{F}_n\}$.

17. **Homework 3.A. (8th Mar)** Let X, Y be two independent $\text{Exp}(\lambda)$ random variables and $Z := X + Y$. Show that for any non-negative measurable h we have $\mathbb{E}\left[h(X)|Z\right] = \frac{1}{Z}\int\limits_0^Z h(t)\,\mathrm{d}t$.

18. **Homework 3.B. (8th Mar)** Let $\xi_1, \xi_2, \ldots$ be a sequence of iid. random variables with $\mathbf{P}(\xi_1 = 1) = \mathbf{P}(\xi_1 = -1) = 1/2$ and define the simple symmetric random walk $S_n = \xi_1 + \cdots + \xi_n$. For the integers k and l, define the hitting times $T_{-k} = \inf\{n : S_n = -k\}$ and $T_l = \inf\{n : S_n = l\}$ and the stopping time given by their minimum $T = \min(T_{-k}, T_l)$.

    (a) Find $\mathbf{E}(S_T)$ by using the optional stopping theorem for the martingale S_n.

    (b) What is $\mathbf{P}(T_{-k} < T_l)$? *Hint:* Note that the random variable S_T can only take two values.

    (c) We have shown previously that $M_n = S_n^2 - n$ is a martingale. Apply the optional stopping theorem for M_n and T and compute $\mathbf{E}(T)$.

19. **Homework 3.C. (8th Mar)** In the casino, a player's winnings per unit stake on game n are ξ_n where $\{\xi\}_{n=1}^{\infty}$ are iid. random variables with $\mathbf{P}(\xi_n = +1) = p$ and $\mathbf{P}(\xi_n = -1) = q$ with $p + q = 1$ and $p > 1/2$. In other words with probability $q < 1/2$ the player loses the stake and with probability p she gets back twice of the stake. Let C_n be the player's stake on game n. We assume that C_n is previsible, that is $C_{n+1} \in \mathcal{F}_n := \sigma\left(\xi_1, \ldots, \xi_n\right)$ for all n. Let Y_n denote the wealth of the player after the nth round. We assume that there is an $\varepsilon > 0$ such that $0 \leq C_n \leq (1 - \varepsilon)Y_{n-1}$. We call $\alpha = p \log p + q \log q + \log 2$ the entropy.

    (a) Define the function
    $$f(x) = p \ln(1 + x) + q \ln(1 - x)$$
    for $x \in [0, 1]$. Show that f is strictly concave. Find $\max_{x \in [0,1]} f(x)$.

    (b) Prove that for any previsible betting strategy C_n, the process $Z_n = \log Y_n - n\alpha$ is a supermartingale. Show that this implies $\mathbf{E}(\log Y_n - \log Y_0) \leq n\alpha$. *Hint:* Introduce $x_n = C_n / Y_{n-1}$ so that $Y_{n+1} = Y_n \cdot (1 + x_{n+1}\xi_{n+1})$. The function f (defined above) appears when calculating $\mathbf{E}(\log(Y_{n+1}) \,|\, \mathcal{F}_n)$.

    (c) Show that there is a betting strategy for which Z_n is a martingale and that $\mathbf{E}(\log Y_n - \log Y_0) = n\alpha$ is achieved. (This is sometimes called the *log-optimal portfolio* in economics.)

Normal distribution

20. Let $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in $\mathbb{R}^2$, $\mathbf{Y} = (Y_1, Y_2)$ and let $a_1, a_2 \in \mathbb{R}$. Find the distribution of $a_1 Y_1 + a_2 Y_2$.

21. Let $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ and let B be a non-singular matrix. Find the distribution of $\mathbf{X} = B \cdot \mathbf{Y}$.

22. **Homework 4.A. (17th Mar)** Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in $\mathbb{R}^n$. Let $\mathbf{X}_1 := (X_1, \ldots, X_p)$ and $\mathbf{X}_2 := (X_{p+1}, \ldots, X_n)$. Let Σ, Σ_1 and Σ_2 the covariance matrices of X, X_1 and X_2 respectively. Prove that $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent if and only if
    $$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.$$

23. **Homework 4.B. (17th Mar)** Construct a random vector (X, Y) such that both X and Y have one-dimensional normal distribution but (X, Y) does not have a bivariate normal distribution in the general sense that it is not an affine transform of a standard bivariate normal vector.

24. Let $\mathbf{X} = (X_1, \dots, X_d)$ be standard multivariate normal distribution. Let Σ be an $n \times n$ positive semi-definite, symmetric matrix and let $\boldsymbol{\mu} \in \mathbb{R}^d$. Prove that there exists an affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$, such that $T(\mathbf{X}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

25. **Homework 4.C. (17th Mar)** Let X be the height of the father and let Y be the height of the son in a sample of father-son pairs. Assume that (X, Y) is bivariate normal. Assume that in inches

$$\mathbf{E}[X] = 68,\ \mathbf{E}[Y] = 69,\ \sigma_X = \sigma_Y = 2,\ \rho = 0.5$$

where ρ is the correlation of (X, Y). Find the conditional distribution of Y given $X = 80$ (6 feet 8 inches).

Martingales

26. **Homework 5.A. (22nd Mar)** There are n white and n black balls in an urn. We pull out all of them one-by-one without replacement. Whenever we pull

- a black ball we have to pay 1\$,
- a white ball we receive 1\$.

Let $X_0 := 0$ and X_i be the amount we gained or lost after the ith ball was pulled. We define

$$Y_i := \frac{X_i}{2n - i} \text{ for } 1 \le i \le 2n - 1, \quad \text{and} \quad Z_i := \frac{X_i^2 - (2n - i)}{(2n - i)(2n - i - 1)} \text{ for } 1 \le i \le 2n - 2.$$

(a) Prove that $Y = (Y_i)$ and $Z = (Z_i)$ are martingales.

(b) Find $\operatorname{Var}(X_i)$.

27. **Homework 5.B. (22nd Mar)** Let $\xi_1, \xi_2, \dots$ be independent standard normal variables. Recall that their moment generating function is $M(\theta) = \mathbf{E}\left[e^{\theta \xi_i}\right] = e^{\theta^2/2}$. Let $a, b \in \mathbb{R}$ and define $S_n = \sum_{k=1}^{n} \xi_k$ and $X_n = e^{aS_n - bn}$. Prove that

(a) $X_n \to 0$ a.s. iff $b > 0$.

(b) $X_n \to 0$ in L^r iff $r < \frac{2b}{a^2}$.

28. **Homework 5.C. (22nd Mar)** Extension of part (iii) of Doob's optional stopping theorem. Let X be a supermartingale. Let T be a stopping time with $\mathbf{E}[T] < \infty$ as in part (iii) of Doob's optional stopping theorem. Assume that there is a C such that

$$\mathbf{E}\left[|X_k - X_{k-1}| | \mathcal{F}_{k-1}\right] \le C$$

holds almost surely for all $k > 0$. Prove that $\mathbf{E}[X_T] \le \mathbf{E}[X_0]$.

29. Let $\mathbf{X}, \mathbf{Z}$ be $\mathbb{R}^d$-valued random variables defined on the $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that

$$\mathbf{E}\left[e^{i\mathbf{t}\cdot\mathbf{X} + i\mathbf{s}\cdot\mathbf{Z}}\right] = \mathbf{E}\left[e^{i\mathbf{t}\mathbf{X}}\right] \cdot \mathbf{E}\left[e^{i\mathbf{s}\mathbf{Z}}\right]$$

for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}^n$. Prove that $\mathbf{X}, \mathbf{Z}$ are independent.

30. Construct a martingale which is not a Markov chain.

31. **Homework 6.A. (31st Mar)** Let X_n be a discrete time birth–death process (that is a Markov chain on the non-negative integers) with transition probabilities

$$p(i,i+1) = p_i, \quad p(i,i-1) = 1 - p_i = q_i$$

for $i \geq 1$ which are all assumed to be strictly positive. We define the function

$$g : \mathbb{N} \to \mathbb{R}^+, \quad g(0) = 0, \quad g(k) = 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \frac{q_i}{p_i}.$$

(a) Prove that $Z_n := g(X_n)$ is a martingale for the natural filtration.

(b) Let $0 < i < n$ be fixed. Find the probability that a process started from i gets to n earlier than to 0.

32. **Homework 6.B. (31st Mar)** Let a_n be a deterministic sequence of real numbers and let $\{\varepsilon_n\}$ be an iid. sequence of random variables satisfying $\mathbb{P}(\varepsilon_n = \pm 1) = \frac{1}{2}$. Show that $\sum\limits_{n=1}^{\infty} \varepsilon_n a_n$ converges almost surely if and only if $\sum\limits_{n=1}^{\infty} a_n^2 < \infty$.

33. **Homework 6.C. (31st Mar)** Let $X_1, X_2, \ldots$ be an independent sequence of random variables with $X_1 = 0$ and for $n \geq 2$ with $\mathbf{P}(X_n = -n^2) = \frac{1}{n^2}$ and $\mathbf{P}\left(X_n = \frac{n^2}{n^2-1}\right) = 1 - \frac{1}{n^2}$. Let $S_n := X_1 + \cdots + X_n$. Show that

(a) $\{S_n\}$ is a martingale with zero expectation which converges to ∞ a.s.

(b) $\lim\limits_{n\to\infty} S_n/n = 1$.

Hint: The Borel–Cantelli lemma can be useful which says that if the events $A_1, A_2, \ldots$ satisfy $\sum_k \mathbf{P}(A_k) < \infty$, then with probability one, only finitely many of them happen.

34. **Homework 7.A. (12th Apr)**

(a) Let $X = (X_n)$ be an L^2 random walk that is a martingale, i.e. let Z_k be iid. random variables with 0 mean and finite variance σ^2 and let $X_n = Z_1 + \cdots + Z_n$. Prove that the angle bracket process of X is $A_n = n\sigma^2$.

(b) Let X_n be an L^2 martingale with angle bracket process A_n. Let C_n be previsible and also L^2. Prove that the angle bracket process of the martingale transform $Y = C \bullet X$ is given by $C^2 \bullet A$.

35. **Homework 7.B. (12th Apr)** Let $M = (M_n)$ be a martingale with $M_0 = 0$ and $|M_k - M_{k-1}| < C$ for a $C \in \mathbb{R}$. Let T be a stopping time which is finite a.s. and define

$$U_n = \sum_{k=1}^{n} (M_k - M_{k-1})^2 \mathbb{1}_{T \geq k}, \quad V_n = 2 \sum_{1 \leq i < j \leq n} (M_i - M_{i-1})(M_j - M_{j-1}) \mathbb{1}_{T \geq j},$$

$$U_\infty = \sum_{k=1}^{\infty} (M_k - M_{k-1})^2 \mathbb{1}_{T \geq k}, \quad V_\infty = 2 \sum_{1 \leq i < j} (M_i - M_{i-1})(M_j - M_{j-1}) \mathbb{1}_{T \geq j}.$$

(a) Prove that $M_{T\wedge n}^2 = U_n + V_n$ and $M_T^2 = U_\infty + V_\infty$.

(b) Assume further that $\mathbf{E}[T^2] < \infty$. Show that $\lim\limits_{n\to\infty} U_n = U_\infty$ a.s. and $\mathbf{E}[U_\infty] < \infty$ and $\mathbf{E}[V_n] = \mathbf{E}[V_\infty] = 0$.

(c) Conclude that if $\mathbf{E}[T^2] < \infty$, then $\lim_{n\to\infty} \mathbf{E}[M^2_{T\wedge n}] = \mathbf{E}[M^2_T]$.

36. **Homework 7.C. (12th Apr)** Wald equalities. Let $Y_1, Y_2, \ldots$ be iid. L^1 random variables. Let $S_n := Y_1 + \cdots + Y_n$ and we write $\mu = \mathbf{E}[Y_i]$. Let T be a stopping time satisfying $\mathbf{E}[T] < \infty$.

    (a) Prove that
    $$\mathbf{E}[S_T] = \mu \cdot \mathbf{E}[T].$$

    (b) Assume further that Y_i are bounded ($\exists C_i \in \mathbb{R}$ with $|Y_i| < C_i$) and $\mathbf{E}[T^2] < \infty$. We write $\sigma^2 = \mathrm{Var}(Y_i)$. Then
    $$\mathbf{E}\left[(S_T - \mu T)^2\right] = \sigma^2 \cdot \mathbf{E}[T].$$

    Hint: Introduce an appropriate martingale and apply the result of the previous exercise.

37. **Homework 8.A. (19th Apr)** A branching process $Z = (Z_n)_{n=0}^\infty$ is defined recursively by a given family of non-negative integer valued iid. random variables $\left\{X_k^{(n)}\right\}_{k,n=1}^\infty$ as follows:
    $$Z_0 := 1, \quad Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}, \quad n \geq 0.$$
    Let $\mu = \mathbf{E}\left[X_k^{(n)}\right]$ and $\mathcal{F}_n = \sigma(Z_0, Z_1, \ldots Z_n)$. We write $f(s)$ for the generating function of the offspring distribution, that is
    $$f(s) = \mathbf{E}\left(s^{X_k^{(n)}}\right) = \sum_{m=0}^\infty \mathbf{P}\left(X_k^{(n)} = m\right) s^m$$
    for any k and n. Further, let
    $$\{\text{extinction}\} = \{Z_n \to 0\} = \{\exists n,\ Z_n = 0\}, \quad \{\text{explosion}\} = \{Z_n \to \infty\}$$
    and denote $q = \mathbf{P}(\text{extinction})$. Recall that q is the smaller (if there are two) fixed point of $f(s)$, that is q is the smallest solution of $f(q) = q$.

    (a) Prove by induction that $\mathbf{E}[Z_n] = \mu^n$.

    (b) Show that $\mathbf{E}\left[s^{Z_{n+1}} | \mathcal{F}_n\right] = f(s)^{Z_n}$ for every $s \geq 0$. Explain that q^{Z_n} is a martingale and $\lim_{n\to\infty} Z_n = Z_\infty$ exists a.s.

    (c) Let $T = \min\{n : Z_n = 0\}$ be the extinction time with $T = \infty$ if $Z_n > 0$ always. Prove by dominated convergence that $q = \mathbf{E}\left[q^{Z_T}\right] = \mathbf{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] + \mathbf{E}\left[q^{Z_T} \cdot \mathbb{1}_{T<\infty}\right]$.

    (d) Prove that $\mathbf{E}\left[q^{Z_\infty} \cdot \mathbb{1}_{T=\infty}\right] = 0$.

    (e) Conclude that if $T(\omega) = \infty$ then $Z_\infty = \infty$, hence
    $$\mathbf{P}(\text{extinction}) + \mathbf{P}(\text{explosion}) = 1.$$

38. **Homework 8.B. (19th Apr)** Assume that for the offspring distribution of the branching process Z_n, we have
    $$\mu = \mathbf{E}\left[X_k^{(n)}\right] < \infty \text{ and } 0 < \sigma^2 = \mathrm{Var}(X_k^{(n)}) < \infty.$$
    Prove that

(a) $M_n = Z_n/\mu^n$ is a martingale for the natural filtration $\mathcal{F}_n$.

(b) Show that $\mathbf{E}\left[Z_{n+1}^2 | \mathcal{F}_n\right] = \mu^2 Z_n^2 + \sigma^2 Z_n$ and conclude that the martingale M_n is bounded in L^2 if and only if $\mu > 1$.

(c) Assume that $\mu > 1$ which implies that $M_\infty = \lim_{n\to\infty} M_n$ exists in L^2 and a.s. Prove that

$$\mathrm{Var}(M_\infty) = \frac{\sigma^2}{\mu(\mu-1)}.$$

39. **Homework 8.C. (19th Apr)** Let $X_1, X_2, \ldots$ be iid. random variables with a continuous distribution function. Let E_i be the event that a record occurs at time n. That is $E_1 = \Omega$ and $E_n = \{X_n > X_m,\ \forall m < n\}$. Prove that $\{E_i\}_{i=1}^\infty$ are independent and $\mathbf{P}(E_i) = 1/i$. *Hint:* For the independence show that for any $i_1 < \cdots < i_n$ one has $\mathbf{P}(E_{i_1} | E_{i_2} \ldots E_{i_n}) = 1/i_1$ and write $\mathbf{P}(E_{i_1} \ldots E_{i_n}) = \prod_{j=1}^n \mathbf{P}(E_{i_j} | E_{i_{j+1}} \ldots E_{i_n})$.

40. Let $E_1, E_2, \ldots$ be independent with $\mathbf{P}(E_i) = 1/i$. Let $Y_i := \mathbb{1}_{E_i}$ and $N_n := Y_1 + \cdots + Y_n$. In the special case of the previous homework, N_n is the number of records until time n.

    (a) Prove that $\sum_{k=2}^{\infty} \frac{Y_k - 1/k}{\log k}$ converges almost surely.

    (b) Using Kronecker's lemma conclude that $\lim_{n\to\infty} \frac{N_n}{\log n} = 1$ a.s.

41. Assume that in the branching process Z_n, $q = 1$ where q is the probability of extinction. Prove that $M_n = Z_n/\mu^n$ is not a uniformly integrable martingale.

42. Let $\mathcal{C}$ be a class of random variables of $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\mathcal{C}$ is L^p bounded for some $p > 1$, that is, $\exists p > 1$ and $A \in \mathbb{R}$ such that $\mathbf{E}[|X|^p] < A$ for all $X \in \mathcal{C}$. Show that then $\mathcal{C}$ is UI.

43. **Homework 9.A (26th Apr)** Let $\mathcal{C}$ be a class of random variables of $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\mathcal{C}$ is dominated by an integrable random variable, that is $\exists Y \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ such that $|X| \le Y$ a.s. $\forall X \in \mathcal{C}$. Show that then $\mathcal{C}$ is UI.

44. Let $\mathcal{C}$ be a class of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. Prove that $\mathcal{C}$ is UI if and only if the following two conditions hold

    (a) $\mathcal{C}$ is L^1-bounded, that is $\sup\{\mathbf{E}[|X|] : X \in \mathcal{C}\} < \infty$ and

    (b) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$F \in \mathcal{F} \text{ and } \mathbf{P}(F) < \delta \Longrightarrow \mathbf{E}[|X| \mathbb{1}_F] < \varepsilon.$$

45. Let $\mathcal{C}$ and $\mathcal{D}$ be UI classes of random variables. Prove that

$$\mathcal{C} + \mathcal{D} = \{X + Y : X \in \mathcal{C} \text{ and } Y \in \mathcal{D}\}$$

is also UI. *Hint:* Use the previous exercise.

46. **Homework 9.B (26th Apr)** Let $\mathcal{C}$ be a UI family of random variables. Let us define

$$\mathcal{D} := \{Y : \exists X \in \mathcal{C},\ \exists \mathcal{G} \text{ sub-}\sigma\text{-algebra of } \mathcal{F} \text{ s.t. } Y = \mathbf{E}[X|\mathcal{G}]\}.$$

Prove that $\mathcal{D}$ is also UI.

47. **Homework 9.C (26th Apr)** Let $X_1, X_2, \dots$ be iid. random variables with $\mathbf{E}[X^+] = \infty$ and $\mathbf{E}[X^-] < \infty$. (Recall $X = X^+ - X^-$ and $X^+, X^- \geq 0$.) Use the SLLN to prove that $S_n/n \to \infty$ a.s. where $S_n := X_1 + \dots + X_n$. *Hint:* For $M > 0$ let $X_i^M = X_i \wedge M$ and $S_n^M = X_n^M + \dots + X_n^M$. Explain why $\lim_{n\to\infty} S_n^M/n = \mathbf{E}[X_i^M]$ and $\liminf_{n\to\infty} S_n/n \geq \lim_{n\to\infty} S_n^M/n$.

48. **Homework 10.A (3rd May)** (Infinite Monkey Theorem) Prove that a monkey typing at random on a typewriter for infine time will type the complete works of Shakespeare eventually.

49. **Homework 10.B (3rd May)** (Azuma–Hoeffding inequality)

    (a) Assume that Y is a random variable which takes values from $[-c, c]$ and $\mathbf{E}[Y] = 0$ holds. Prove that for all $\theta \in \mathbb{R}$ we have
    $$\mathbf{E}\left[e^{\theta Y}\right] \leq \cosh(\theta c) \leq \exp\left(\frac{1}{2}\theta^2 c^2\right).$$
    Hint: Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then by the convexity of f we have
    $$f(y) \leq \frac{c-y}{2c} f(-c) + \frac{c+y}{2c} f(c).$$

    (b) Let M be a martingale with $M_0 = 0$ such that for a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ we have $|M_n - M_{n-1}| \leq c_n$ for all n. Then the following inequality holds for all $x > 0$:
    $$\mathbf{P}\left(\sup_{k \leq n} M_k \geq x\right) \leq \exp\left(-\frac{1}{2}x^2 / \sum_{k=1}^{n} c_k^2\right).$$
    Hint: Use submartingale inequlaity as in the proof of LIL. Then present M_n (in the exponent) like a telescopic sum of its increments. Use the orthogonality of martingale increments. Use the first part of this exercise and find the minimum in θ of the expression in the exponent.

50. **Homework 10.C (3rd May)** m balls are placed one by one into n urns independently at random with uniform distribution among the urns. Let $\mathcal{F}_k$ be the σ-algebra generated by the place of the first k balls and define the martingale $M_k = \mathbf{E}(N|\mathcal{F}_k)$ where N is the number of empty urns after placing the last ball. Let Y_k be the number of empty urns after placing the kth ball. Show that

    (a) $M_k = Y_k \left(\frac{n-1}{n}\right)^{m-k}$;

    (b) $|M_k - M_{k-1}| \leq \left(\frac{n-1}{n}\right)^{m-k}$;

    (c)
    $$\mathbf{P}(|N - \mu| \geq \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2(n-1/2)}{n^2 - \mu^2}\right)$$
    where $\mu = \mathbf{E}N = n\left(\frac{n-1}{n}\right)^m$.

    Hint: For part (b) note that the possible values of Y_k are Y_{k-1} or $Y_{k-1} - 1$. For part (c) use the Azuma–Hoeffding inequality for the martingales $M_k - \mu$ and $-M_k + \mu$.

51. (Exercise for Markov chain CLT) Consider the following Markov chain $\{X_n\}_{n=0}^{\infty}$. The state space is $\mathbb{Z}$. The transition probabilities are $p(0,1) = p(0,-1) = \frac{1}{2}$ and for an arbitrary $x \in \mathbb{N}^+$ we have
$$p(x, x+1) = p(x, 0) = \frac{1}{2}, \qquad p(-x, -x-1) = p(-x, 0) = \frac{1}{2}.$$

(a) Find the stationary measure π for X_n.

(b) Define the operator $P : L^1(\mathbb{Z}, \pi) \to L^1(\mathbb{Z}, \pi)$ by $(Pg)(i) := \sum_{j \in \mathbb{Z}} p(i,j)g(j)$ and let I be the identity on $L^1(\mathbb{Z}, \pi)$, i.e. P acts on g as a multiplication of an infinite matrix and an infinite vector. Further, let $f : \mathbb{Z} \to \mathbb{R}$ be an arbitrary function satisfying the following conditions:

$$\forall x \in \mathbb{Z},\ f(x) = -f(-x),\ \text{and } \exists a < \sqrt{2} \text{ s.t. } f(x) < a^{|x|} \text{ for all } x \text{ large enough.}$$

For example polynomials of the form $f(x) = \sum_{i=1}^{n} b_{2i-1} x^{2i-1}$.

(c) Construct a function $g \in L^2(\pi)$ such that $((I - P) \cdot g)(i) = f(i)$.

(d) From now on we always assume that $f(x) = x^{-3}$. Determine

$$\sigma^2 := \mathbf{E}_\pi \left[(g(X_1) - \mathbf{E}(g(X_1)|\mathcal{F}_0))^2 \right].$$

(e) Prove that $\mathbf{P}\left(-3\sigma\sqrt{n} \leq f(X_1) + \cdots + f(X_n) \leq 3\sigma\sqrt{n}\right) \geq 0.99$ for sufficiently large n.