# Markov processes and martingales
# 2nd resit midterm test

10th Dec 2018

1. (a) (2 points) Define

$$f(x) = \begin{cases} \frac{1}{2x^2} & \text{if } x \geq 1, \\ 0 & \text{if } -1 < x < 1, \\ -\frac{1}{x^3} & \text{if } x \leq -1. \end{cases}$$

   Check that $f(x)$ is the density function of a probability distribution.

   (b) (3 points) Let $X$ be a random variable with density $f(x)$. Let it be decomposed as $X = X^+ - X^-$ with $X^+, X^- \geq 0$. Show that $\mathbf{E}(X^+) = \infty$ and $\mathbf{E}(X^-) < \infty$.

   (c) (4 points) Let $X_1, X_2, \ldots$ be an iid. sequence of random variables with density $f(x)$ and define $S_n = X_1 + \cdots + X_n$. Use the SLLN to prove that $S_n/n \to \infty$ a.s. in the following steps. For any $M > 0$, define $X_k^M = \min(X_k, M)$ and $S_n^M = X_1^M + \cdots + X_n^M$. Explain why $S_n^M/n \to \mathbf{E}(X_1^M)$ and $\liminf_{n\to\infty} S_n/n \geq \lim_{n\to\infty} S_n^M/n$.

2. (a) (4 points) Let $Y_1, Y_2, \ldots$ be iid. $L^1$ random variables. Let $S_n := Y_1 + \cdots + Y_n$ and we write $\mu = \mathbf{E}(Y_1)$. Let $T$ be a stopping time satisfying $\mathbf{E}(T) < \infty$. Prove that

$$\mathbf{E}\left[S_T\right] = \mu \cdot \mathbf{E}\left[T\right].$$

   *Hint:* Decompose the probability space into the the disjoint union of the events $\{T = k\}$ for $k = 1, 2, \ldots$ and use dominated convergence to verify the exchange of the expectation with the infinite summation.

   (b) (4 points) Let $X_1, X_2, \ldots$ be an iid. sequence of random variables with distribution $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$ and define the simple random walk $Z_n = X_1 + \cdots + X_n$. Let $U = \min\{n > 0 : Z_n = 1\}$ be the first hitting time of 1. Use the first part of this exercise to conclude that $\mathbf{E}(U)$ cannot be finite.