Markov processes and martingales
1st resit midterm test

10th Dec 2018

1. (a) (4 points) Let $M_n^{(1)}$ and $M_n^{(2)}$ be two martingales with respect to the same filtration $\mathcal{F}_n$. Prove that $M_n = \max(M_n^{(1)}, M_n^{(2)})$ is a submartingale with respect to $\mathcal{F}_n$.

   (b) (2 points) Let $X_1, X_2, \dots$ be an iid. sequence of random variables with $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$ and let $Y_1, Y_2, \dots$ be another iid. sequence with $\mathbf{P}(Y_1 = 1) = 2/3$ and $\mathbf{P}(Y_1 = -1) = 1/3$. Assume that the two sequences are independent of each other. Show that $N_n^{(1)} = X_1 + \cdots + X_n$ and $N_n^{(2)} = Y_1 + \cdots + Y_n - n/3$ define two martingales with respect to the natural filtration.

   (c) (3 points) By the first part of the exercise, $N_n = \max(N_n^{(1)}, N_n^{(2)})$ is a submartingale. What is the distribution of N_1? Compute $\mathbf{E}(N_1)$ and check that it is non-negative.

2. Let $\mathbf{X} = (X_1, X_2)$ be a standard bivariate normal vector. Define $\mathbf{Y} = A\mathbf{X} + \underline{b}$ where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

   (a) (2 points) What is the bivariate density function of the random vector $\mathbf{Y} = (Y_1, Y_2)$?

   (b) (2 points) What is the covariance of the two components Y_1 and Y_2?

   (c) (2 points) What is the expectation of $Y_1 + Y_2$?

   (d) (2 points) What is the variance of $Y_1 + Y_2$?