Markov processes and martingales
2nd midterm test

5th Dec 2018

1. Let $(Z_n)_{n=0}^{\infty}$ be a branching process defined recursively by the iid. random variables $\{X_k^{(n)}\}_{k,n=1}^{\infty}$ with distribution $\mathbf{P}(X_1^{(1)} = 0) = 1/4$ and $\mathbf{P}(X_1^{(1)} = 2) = 3/4$. Let

$$Z_0 := 1, \quad Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}, \quad n \geq 0.$$

   (a) (3 points) Compute the generating function of the offspring distribution given by

$$f(s) = \mathbf{E}\left(s^{X_1^{(1)}}\right) = \sum_{m=0}^{\infty} \mathbf{P}\left(X_1^{(1)} = m\right) s^m.$$

   Further, let

$$\{\text{extinction}\} = \{Z_n \to 0\} = \{\exists n,\ Z_n = 0\}, \quad \{\text{explosion}\} = \{Z_n \to \infty\}.$$

   Recall that $q = \mathbf{P}\,(\text{extinction})$ is the smaller (if there are two) fixed point of $f(s)$. Find the value of q.

   (b) (3 points) Let $\mathcal{F}_n = \sigma(Z_0, Z_1, \ldots Z_n)$. Show that $\mathbf{E}\left[s^{Z_{n+1}} | \mathcal{F}_n\right] = f(s)^{Z_n}$ for every $s \geq 0$. Explain that q^{Z_n} is a martingale and $\lim_{n\to\infty} Z_n = Z_\infty$ exists a.s.

   (c) (3 points) Let $T = \min\{n : Z_n = 0\}$ be the extinction time with $T = \infty$ if $Z_n > 0$ always. Prove by dominated convergence that $q = \mathbf{E}\left[q^{Z_T}\right]$.

2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where $\Omega = [0, 1]$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbf{P}$ is the Lebesgue measure. Then the random variable $U(\omega) = \omega$ has uniform distribution on $[0, 1]$. Define the sequence of random variables

$$X_n = \frac{1}{U^\alpha} \mathbb{1}_{\{U \in [1/n, 1]\}}$$

   where $\alpha > 0$ is a parameter.

   (a) (2 points) Show that for $\alpha \in (0, 1)$, the random variables X_n are uniformly integrable. *Hint:* One can show that they are dominated by an L^1 random variable.

   (b) (2 points) Prove that for $\alpha \geq 1$, the random variables X_n are not uniformly integrable by computing $\mathbf{E}(X_n)$ and by showing that it diverges to infinity as $n \to \infty$.

   (c) (2 points) What is the almost sure limit X of X_n for any $\alpha > 0$?

   (d) (2 points) For which values of α does the sequence X_n converge also in L^1 to the almost sure limit X?