# Markov processes and martingales
# 1st midterm test

17th Oct 2018

1. One water lily appears in a pond. If the weather conditions are favorable, the number of water lilies doubles for the next year, if the weather is unfavorable, the water lily disappears. The procedure repeats in each year as follows. Under good conditions, the number of lilies in the pond doubles for the next year. Under bad conditions, the number of lilies halves for next year (if there was just one piece, it disappears and does not reappear). The weather conditions change in every year randomly, they are good or bad with equal probability independently of each other.

   (a) (2 points) What is the probability that there are exactly four pieces of water lilies in the pond after four years?

   (b) (2 points) Show that the logarithm of the number of water lilies in the pond observed in each year forms a martingale with respect to its natural filtration.

   (c) (4 points) What is the probability that the number of water lilies becomes 32 before they disappear from the pond? *Hint:* Use the optional stopping theorem with an appropriate stopping time.

2. Let $\mathbf{X} = (X_1, X_2)$ have bivariate normal distribution with mean vector $(2, -3)$ and covariance matrix
$$\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}.$$

   (a) (2 points) What is the correlation of the two components $X_1$ and $X_2$?

   (b) (2 points) Let $A$ be a non-singular $2 \times 2$ matrix and $\underline{b}$ be a vector in $\mathbb{R}^2$. What is the distribution of $A\mathbf{X} + \underline{b}$? (Normality does not have to be proved.)

   (c) (3 points) Find a linear transformation which maps $\mathbf{X}$ to standard bivariate normal distribution?