Markov processes and martingales exam

18th Jan 2019

Theoretical part

1. (a) (2 points) Describe the monkey at the typewriter problem.
   (b) (7 points) Compute the expected number of letters for the appearance of the word ABRACADABRA. You may use Doob's optional stopping theorem without proof.
2. (a) (2 points) For a discrete time stochastic process define the number of upcrossings of an interval.
   (b) (3 points) State and prove Doob's upcrossing lemma for the number of upcrossings for a supermartingale.
   (c) (4 points) State and prove Doob's forward convergence theorem.
3. (a) (3 points) State the central limit theorem for martingales without proof.
   (b) (6 points) State and prove the central limit theorem for Markov chains using the central limit theorem for martingales.

Exercise part

4. There are n white and n black balls in an urn. We pull out all of them one-by-one without replacement. Whenever we pull
   - a black ball we have to pay 1$,
   - a white ball we receive 1$.

   Let $X_0 := 0$ and X_i be the amount we gained or lost after the ith ball was pulled. We define

   $$Y_i := \frac{X_i}{2n-i} \text{ for } 1 \leq i \leq 2n-1, \quad \text{and} \quad Z_i := \frac{X_i^2 - (2n-i)}{(2n-i)(2n-i-1)} \text{ for } 1 \leq i \leq 2n-2.$$

   (a) (6 points) Prove that $Y = (Y_i)$ and $Z = (Z_i)$ are martingales.
   (b) (3 points) Find $\mathrm{Var}(X_i)$.
5. Let X_n be a discrete time birth–death process with transition probabilities

   $$p(i, i+1) = \frac{1}{i+1}, \quad p(i, i-1) = \frac{i}{i+1}$$

   for $i \geq 1$. These probabilities can be understood as follows. In state i, any of the i individuals can disappear or a new one can appear with equal probabilities. We define the function

   $$g : \mathbb{N} \to \mathbb{R}^+, \quad g(0) = 0, \quad g(k) = 1 + \sum_{j=1}^{k-1} j! \quad k = 1, 2, \ldots.$$

   (a) (4 points) Prove that $Z_n := g(X_n)$ is a martingale for the natural filtration.
   (b) (5 points) Evaluate the function $g(k)$ for $k = 0, 1, 2, \ldots, 5$ and find the probability that a process started from 1 gets to 5 earlier than to 0.

6. m balls are placed one by one into n urns independently at random with uniform distribution among the urns. Let $\mathcal{F}_k$ be the σ-algebra generated by the place of the first k balls and define the martingale $M_k = \mathbf{E}(N|\mathcal{F}_k)$ where N is the number of empty urns after placing the last ball. Let Y_k be the number of empty urns after placing the kth ball. Show that

   (a) (2 points) $M_k = Y_k \left(\frac{n-1}{n}\right)^{m-k}$;

   (b) (2 points) $|M_k - M_{k-1}| \le \left(\frac{n-1}{n}\right)^{m-k}$;

   (c) (3 points)
   $$\mathbf{P}(|N-\mu| \ge \varepsilon) \le 2\exp\left(-\frac{\varepsilon^2(n-1/2)}{n^2-\mu^2}\right)$$
   where $\mu = \mathbf{E}N = n\left(\frac{n-1}{n}\right)^m$. *Hint:* Use the Azuma–Hoeffding inequality for the martingales $M_k - \mu$ and $-M_k + \mu$. It says that if $|X_n - X_{n-1}| \le c_n$ for all n for a martingale X_n with $X_0 = 0$, then
   $$\mathbf{P}\left(\sup_{k\le n} X_k \ge x\right) \le \exp\left(-\frac{1}{2}x^2 / \sum_{k=1}^{n} c_k^2\right).$$

   (d) (2 points) Assume that $m = n$ and they are large, i.e. $n \to \infty$. How much is μ asymptotically? Use the previous part of the exercise for $\varepsilon = K\sqrt{n}$ to show that the upper bound on the probability $\mathbf{P}(|N-\mu| \ge \varepsilon)$ has a limit in $(0,1)$.