Markov processes and martingales exam

4th Jan 2019

Theoretical part

1. (a) (2 points) Describe what Pólya's urn model is.
   (b) (7 points) Prove that the ratio of one of the colors is a martingale and it converges. In the case when the limit is the uniform distribution, identify this limit distribution with computations.

2. (9 points) Suppose that $X_1, X_2, \ldots$ are independent random variables with zero mean and variance $\sigma_k^2 = \mathrm{Var}(X_k)$. State and prove the theorem about the relation between the finiteness of $\sum_{k=1}^{\infty} \sigma_k^2$ and the almost sure convergence of $\sum_{k=1}^{\infty} X_k$. The Pythagorean formula, the L^2 convergence theorem for martingales and the fact that a stopped martingale is a martingale can be used without proof.

3. (a) (2 points) Define the Doob martingale, i.e. the conditional expectation of a single random variable with respect to a filtration. Show that it is a martingale.
   (b) (7 points) State and prove Lévy's upward theorem. The uniform integrability does not have to be proved. For the identification of the almost sure and L^1 limit to be the appropriate conditional expectation, Doob's forward convergence theorem can be used.

Exercise part

4. Martin Gaal the Slovak gambler with Hungarian origin plays in the casino. His winnings per unit stake on game n are ξ_n where $\{\xi\}_{n=1}^{\infty}$ are iid. random variables with $\mathbf{P}(\xi_n = +1) = 0.6$ and $\mathbf{P}(\xi_n = -1) = 0.4$. In other words with probability 0.4 Martin loses the stake and with probability 0.6 he gets back twice of the stake. Let C_n be Martin's stake on game n which is assumed to be previsible. Let Y_n denote Martin's wealth after the nth round. We assume that $0 \leq C_n \leq Y_{n-1}$. We call $\alpha = 0.6 \log 0.6 + 0.4 \log 0.4 + \log 2$ the entropy.

   (a) (3 points) Define the function
   $$f(x) = 0.6 \ln(1+x) + 0.4 \ln(1-x)$$
   for $x \in [0,1]$. Show that f is strictly concave. Find $\max_{x \in [0,1]} f(x)$.

   (b) (3 points) Prove that for any previsible betting strategy C_n, the process $Z_n = \log Y_n - n\alpha$ is a supermartingale. Show that this implies $\mathbf{E}(\log Y_n - \log Y_0) \leq n\alpha$. *Hint:* Introduce $x_n = C_n / Y_{n-1}$ so that $Y_{n+1} = Y_n \cdot (1 + x_{n+1}\xi_{n+1})$. The function f (defined above) appears when calculating $\mathbf{E}(\log(Y_{n+1}) \mid \mathcal{F}_n)$.

   (c) (3 points) Show that there is a betting strategy for which Z_n is a martingale and that $\mathbf{E}(\log Y_n - \log Y_0) = n\alpha$ is achieved.

5. Let $M = (M_n)$ be a martingale with $M_0 = 0$ and $|M_k - M_{k-1}| < C$ for a $C \in \mathbb{R}$. Let T be a stopping time which is finite a.s. and define
   $$U_n = \sum_{k=1}^{n} (M_k - M_{k-1})^2 \mathbb{1}_{T \geq k}, \quad V_n = 2 \sum_{1 \leq i < j \leq n} (M_i - M_{i-1})(M_j - M_{j-1}) \mathbb{1}_{T \geq j},$$
   $$U_\infty = \sum_{k=1}^{\infty} (M_k - M_{k-1})^2 \mathbb{1}_{T \geq k}, \quad V_\infty = 2 \sum_{1 \leq i < j} (M_i - M_{i-1})(M_j - M_{j-1}) \mathbb{1}_{T \geq j}.$$

   (a) (3 points) Prove that $M_{T \wedge n}^2 = U_n + V_n$ and $M_T^2 = U_\infty + V_\infty$.

   (b) (4 points) Assume further that $\mathbf{E}\left[T^2\right] < \infty$. Show that $\lim_{n \to \infty} U_n = U_\infty$ a.s. and $\mathbf{E}\left[U_\infty\right] < \infty$ and $\mathbf{E}\left[V_n\right] = \mathbf{E}\left[V_\infty\right] = 0$.

   (c) (2 points) Conclude that if $\mathbf{E}\left[T^2\right] < \infty$, then $\lim_{n \to \infty} \mathbf{E}\left[M_{T \wedge n}^2\right] = \mathbf{E}\left[M_T^2\right]$.

6. (a) (4 points) Let $\mathcal{C}$ be a class of random variables of $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\mathcal{C}$ is L^p bounded for some $p > 1$, that is, $\exists p > 1$ and $A \in \mathbb{R}$ such that $\mathbf{E}[|X|^p] < A$ for all $X \in \mathcal{C}$. Show that then $\mathcal{C}$ is uniformly integrable.

   (b) (5 points) Let $\varepsilon > 0$ be fixed. Let X_n be a random variable with density

$$f_n(x) = \begin{cases} \frac{c_{\varepsilon,n}}{|x|^{2+\varepsilon}} & \text{if } 1 \leq |x| \leq n \\ 0 & \text{otherwise} \end{cases}$$

   Compute the value of $c_{\varepsilon,n}$. Using the previous part of the exercise show that the sequence $(X_n)_{n=1}^{\infty}$ is uniformly integrable.