Markov processes and martingales exam

19th Dec 2018

Theoretical part

1. (a) (2 points) Define what a stopping time is with respect to a filtration.
   (b) (7 points) State and prove the theorem about the supermartingale property of stopped supermartingales. (Other results used in this proof also need to be proved.)
2. (9 points) State and prove the Pythagorean formula for L^2 martingales and the convergence result about L^2 bounded martingales.
3. (a) (2 points) What does ergodicity of a dynamical system mean in terms of invariant sets?
   (b) (2 points) Describe the rotation of the circle.
   (c) (5 points) State and prove the necessary and sufficient condition learnt on the class for ergodicity in the rotation of the circle.

Exercise part

4. (9 points) Let X_j, $j = 1, 2, \dots$ be iid. random variables with common distribution $\mathbf{P}(X_j = 1) = 2/3$ and $\mathbf{P}(X_j = -1) = 1/3$ and let $S_n = X_1 + \dots + X_n$. Compute the conditional expectations $\mathbf{E}(X_1|\sigma(S_n))$, $\mathbf{E}(S_n|\sigma(X_1))$ and $\mathbf{E}(S_{n+m}^2|\sigma(S_n))$.
5. Let $\xi_1, \xi_2, \dots$ be independent standard normal variables. Recall that their moment generating function is $M(\theta) = \mathbf{E}\left[e^{\theta\xi_i}\right] = e^{\theta^2/2}$. Let $a, b \in \mathbb{R}$ and define $S_n = \sum_{k=1}^{n} \xi_k$ and $X_n = e^{aS_n - bn}$. Prove that
   (a) (3 points) $X_n \to 0$ a.s. iff $b > 0$;
   (b) (3 points) $X_n \to 0$ in L^r iff $r < \frac{2b}{a^2}$;
   (c) (3 points) X_n is uniformly integrable iff $2b < a^2$.*
6. Azuma–Hoeffding inequality
   (a) (4 points) Assume that Y is a random variable which takes values from $[-c, c]$ and $\mathbf{E}[Y] = 0$ holds. Prove that for all $\theta \in \mathbb{R}$ we have
   $$\mathbf{E}\left[e^{\theta Y}\right] \le \cosh(\theta c) \le \exp\left(\frac{1}{2}\theta^2 c^2\right).$$
   Hint: Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then by the convexity of f we have
   $$f(y) \le \frac{c-y}{2c} f(-c) + \frac{c+y}{2c} f(c).$$
   (b) (5 points) Let M be a martingale with $M_0 = 0$ such that for a sequence of positive numbers $\{c_n\}_{n=1}^{\infty}$ we have $|M_n - M_{n-1}| \le c_n$ for all n. Then the following inequality holds for all $x > 0$:
   $$\mathbf{P}\left(\sup_{k\le n} M_k \ge x\right) \le \exp\left(-\frac{1}{2}x^2 \Big/ \sum_{k=1}^{n} c_k^2\right).$$
   Hint: Use submartingale inequlaity as in the proof of LIL. Then present M_n (in the exponent) like a telescopic sum, then use the orthogonality of martingale increments. Use the first part of this exercise and find the minimum in θ of the expression in the exponent.

*The true condition of uniform integrability in this exercise is $2b > a^2$ or $a = b = 0$.