Markov chains and stopping times

Prop

Let X_n be an irreducible MC on S. Let ν be a prob. measure on S.
Let U be a stopping time with $\mathbb{E}(U) < \infty$ s.t. $\mathbb{P}_\nu(X_U = x) = \nu(x)$.

Then
$$\frac{\mathbb{E}_\nu\left(\sum_{s=0}^{U-1} \mathbb{1}_{\{X_s = y\}}\right)}{\mathbb{E}_\nu(U)} = \pi(y)$$

Proof let $r(x) = \mathbb{E}_\nu\left(\sum_{s=0}^{U-1} \mathbb{1}_{\{X_s = x\}}\right)$, we show $\sum_{y \in S} r(y) P_{yx} = r(x)$

$$r(x) = \sum_{s=0}^{\infty} \mathbb{P}_\nu(X_s = x,\ s < U) = \mathbb{P}_\nu(X_0 = x) + \sum_{s=1}^{\infty} \mathbb{P}_\nu(X_s = x,\ s < U)$$
$$= \sum_{s=1}^{\infty} \mathbb{P}_\nu(X_s = x,\ s < U) + \mathbb{P}_\nu(X_U = x)$$
$$= \mathbb{E}_\nu\left(\sum_{s=1}^{U} \mathbb{1}_{\{X_s = x\}}\right)$$

$$\sum_{y \in S} r(y) P_{yx} = \sum_{y \in S} \mathbb{E}_\nu\left(\sum_{s=0}^{U-1} \mathbb{1}_{\{X_s = y\}} \mathbb{1}_{\{X_{s+1} = x\}}\right)$$

Applications

$R = \min\{s \geq 1 : X_s = X_0\}$

$T_x = \min\{s \geq 0 : X_s = x\}$

1.) $\mathbb{E}_x(R) = \frac{1}{\pi(x)}$; $\mathbb{E}_x\left(\sum_{s=0}^{R-1} \mathbb{1}_{\{X_s = y\}}\right) = \frac{\pi(y)}{\pi(x)}$ $\qquad \nu = \delta_x,\ U = R$

2.) $x \neq y$ $\quad \mathbb{E}_y\left(\sum_{s=0}^{T_x - 1} \mathbb{1}_{\{X_s = y\}}\right) = \pi(y)\left(\mathbb{E}_x(T_y) + \mathbb{E}_y(T_x)\right)$

Proof: $\nu = \delta_y$ $\quad U = \min\{s > T_x : X_s = y\}$

~~and~~ $\quad \sum_{s=0}^{U-1} \mathbb{1}_{\{X_s = y\}} = \sum_{s=0}^{T_x - 1} \mathbb{1}_{\{X_s = y\}}$

3.) $x, y, z \in S$ different states

$$\mathbb{E}_x\left(\sum_{s=0}^{T_z - 1} \mathbb{1}_{\{X_s = y\}}\right) = \pi(y)\left(\mathbb{E}_x(T_z) + \mathbb{E}_z(T_y) - \mathbb{E}_x(T_y)\right)$$

Proof: $\nu = \delta_x$ $\quad U = \min\{s > 0 : \exists\, 0 < u < v < s : X_u = z, X_v = y, X_s = x\}$

$\mathbb{E}_\nu(U) = \mathbb{E}_x(T_z) + \mathbb{E}_z(T_y) + \mathbb{E}_y(T_x)$

$$\underbrace{\mathbb{E}_\nu\left(\sum_{s=0}^{U-1} \mathbb{1}_{\{X_s = y\}}\right)}_{\pi(y)\,\mathbb{E}_\nu(U)} = \mathbb{E}_x\left(\sum_{s=0}^{T_z - 1} \mathbb{1}_{\{X_s = y\}}\right) + 0 + \underbrace{\mathbb{E}_y\left(\sum_{s=0}^{T_x - 1} \mathbb{1}_{\{X_s = y\}}\right)}_{\pi(y)\left(\mathbb{E}_x(T_y) + \mathbb{E}_y(T_x)\right)}$$

4.) $$\mathbb{P}_x(T_y < T_z) = \frac{\mathbb{E}_x(T_z) + \mathbb{E}_z(T_y) - \mathbb{E}_x(T_y)}{\mathbb{E}_z(T_y) + \mathbb{E}_y(T_z)}$$

Proof : by the strong Markov property

$$\underbrace{\mathbb{E}_x\left(\sum_{s=0}^{T_z-1} \mathbb{1}_{\{X_s=y\}}\right)}_{3.)\ \pi(y)(\mathbb{E}_x(T_z)+\mathbb{E}_z(T_y)-\mathbb{E}_x(T_y))} = \mathbb{P}_x(T_y < T_z)\,\underbrace{\mathbb{E}_y\left(\sum_{s=0}^{T_z-1} \mathbb{1}_{\{X_s=y\}}\right)}_{2.)\ \pi(y)(\mathbb{E}_z(T_y)+\mathbb{E}_y(T_z))}$$

Reversible Markov chains

Let X_n be a Markov chain with transition matrix P started from its stationary distribution π.

Then for any N, the reversed chain $X_N, X_{N-1}, \dots, X_1, X_0$ is also a Markov chain, that is, $\mathbb{P}(X_n = x \mid X_N, X_{N-1} \dots X_{n+1}) = \mathbb{P}(X_n = x \mid X_{n+1})$.

Hint for the proof of Markovity: check that $\mathbb{P}(X_n=x_n, X_{n+2}=x_{n+2} \dots X_N = x_N \mid X_{n+1}=x_{n+1}) = \mathbb{P}(X_n=x_n \mid X_{n+1}=x_{n+1})\,\mathbb{P}(X_{n+2}=x_{n+2} \dots X_N=x_N \mid X_{n+1}=x_{n+1})$
i.e. past and future are conditionally indep. given the present

Prop

The transition matrix of the reversed chain is given by

$$P^*_{xy} = \frac{\pi(y) P_{yx}}{\pi(x)}$$

Further, the operator P^* is the adjoint of P in $\ell^2(S,\pi)$.

Proof : let $P^*_{xy} = \mathbb{P}(X_0 = y \mid X_1 = x)$

Then for any $f, g : S \to \mathbb{R}$ in $\ell^2(S,\pi)$,

$$\langle Pf, g\rangle_\pi = \mathbb{E}_\pi\left(\mathbb{E}(f(X_1)\mid X_0)\cdot g(X_0)\right) = \mathbb{E}_\pi\left(f(X_1)\cdot g(X_0)\right)$$

$$= \mathbb{E}_\pi\left(f(X_1)\,\mathbb{E}(g(X_0)\mid X_1)\right) = \langle f, P^* g\rangle_\pi$$

On the other hand, $\langle Pf, g\rangle_\pi = \sum_{x\in S} \pi(x)\, Pf(x) g(x) = \sum_{x\in S}\sum_{y\in S} \pi(x) P_{xy} f(y) g(x)$

Similarly, $\langle f, P^*g\rangle = \sum_{y\in S} \pi(y) f(y) (P^*g)(y) = \sum_{y\in S}\sum_{x\in S} \pi(y) f(y) P^*_{yx} g(x)$

Def

A Markov chain is reversible if $P^* = P$ holds for its transition matrix, i.e. the detailed balance condition $\pi(x) P_{xy} = \pi(y) P_{yx}$ holds.

Corollary

The Markov chain is reversible if for any $x_0, x_1, \dots, x_n = x_0$, $P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} =$
$= P_{x_0x_{n-1}} P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}$. As a consequence $\mathbb{E}_{x_0}(T_{x_1}) + \mathbb{E}_{x_1}(T_{x_2}) + \dots + \mathbb{E}_{x_{n-1}}(T_{x_0})$
$= \mathbb{E}_{x_0}(T_{x_{n-1}}) + \mathbb{E}_{x_{n-1}}(T_{x_{n-2}}) + \dots + \mathbb{E}_{x_1}(T_{x_0})$.

Random walks on weighted graphs

Let $G=(V,E)$ be a finite graph. Assume that for any edge $(x,y)\in E$, the $w_{xy}=w_{yx}>0$ edge weight is associated. Let $w_x=\sum_{y\in V} w_{xy}$ and $w=\sum_{x\in V} w_x$.

Prop

The transition matrix $P_{xy}=\frac{w_{xy}}{w_x}$ and the stationary distribution $\pi(x)=\frac{w_x}{w}$ defines a reversible Markov chain on the graph G.

Any reversible Markov chain with state space V is of this form with $w_{xy}=w_{yx}=\pi(x)P_{xy}$ (with some $w_{xy}=0$ allowed).

Proof: $\pi(x)P_{xy}=\frac{w_x}{w}\cdot\frac{w_{xy}}{w_x}=\frac{w_{xy}}{w}=\pi(y)P_{yx}$

Random walks and electric networks

Let $G=(V,E)$ be a finite graph with given edge weights $w_{xy}>0$ as above.
Let $\alpha,\beta\in V$ be two different vertices.
Consider the electric network between α and β of the graph which arises by replacing every edge with weight w_{xy} by a resistance $r_{xy}=\frac{1}{w_{xy}}$.

Def

A potential is a function $u:V\to\mathbb{R}$.

A current $j:V\times V\to\mathbb{R}$ with $j(x,y)=0$ if $(x,y)\notin E$, $j(y,x)=-j(x,y)$ for any $(x,y)\in E$ and for any $x\neq\alpha,\beta$ $\sum_{y\in V} j(x,y)=0$ (Kirchhoff). Let $j_{in}=\sum_{y\in V} j(\alpha,y)$ and $j_{out}=\sum_{y\in V} j(y,\beta)$.

Ohm's law

For any $(x,y)\in E$: $u(x)-u(y)=r_{xy}\cdot j(x,y)$.

Ohm problem 1

Given the graph G with resistances r_{xy} and $j_{in}=j_{out}=1$ what are the potential and current functions? Here the effective resistance is $r_{eff}=u(\alpha)-u(\beta)$.

Ohm problem 2

Given the graph G with resistances r_{xy} and $u(\alpha)=1$ and $u(\beta)=0$ what are the potential and current functions? Here the effective resistance is $r_{eff}=\frac{1}{j_{in}}$.

Observation

Given the weighted graph G with X_n a random walk on it,

$$j(x,y) = \mathbb{E}_\alpha\left(\sum_{n=0}^{T_\beta - 1} \mathbb{1}_{\{X_n = x, X_{n+1} = y\}} - \sum_{n=0}^{T_\beta - 1} \mathbb{1}_{\{X_n = y, X_{n+1} = x\}}\right)$$

defines a unit current on G, that is, $j_{in} = j_{out} = 1$,

$u(x) = \mathbb{P}_x(T_\alpha < T_\beta)$ defines a potential on G with $u(\alpha) = 1$ and $u(\beta) = 0$.

Theorem

1) $r_{eff} = \dfrac{\mathbb{E}_\alpha(T_\beta) + \mathbb{E}_\beta(T_\alpha)}{w}$

2) $j(x,y)$ and $u(x)\, r_{eff}$ defined above solves Ohm problem 1.

2) $\dfrac{j(x,y)}{r_{eff}}$ and $u(x)$ defined above solves Ohm problem 2.

Proof: it is enough to show for any $x, y \in V$ that

$u(x) - u(y) = \dfrac{r_{xy}}{r_{eff}}\, j(x,y)$ with r_{eff} replaced by the RHS of 1)

$$j(x,y) = \mathbb{E}_\alpha\left(\sum_{n=0}^{T_\beta - 1} \mathbb{1}_{\{X_n = x\}}\right) P_{xy} - \mathbb{E}_\alpha\left(\sum_{n=0}^{T_\beta - 1} \mathbb{1}_{\{X_n = y\}}\right) P_{yx}$$

$$= \Big(\mathbb{E}_\alpha(T_\beta) + \mathbb{E}_\beta(T_x) - \mathbb{E}_\alpha(T_x)\Big)\pi(x) P_{xy} - \Big(\mathbb{E}_\alpha(T_\beta) + \mathbb{E}_\beta(T_y) - \mathbb{E}_\alpha(T_y)\Big)\pi(y) P_{yx}$$

$$= \Big(\mathbb{E}_\beta(T_x) - \mathbb{E}_\alpha(T_x) - \mathbb{E}_\beta(T_y) + \mathbb{E}_\alpha(T_y)\Big)\frac{w_{xy}}{w}$$

$$u(x) - u(y) = \frac{\mathbb{E}_x(T_\beta) - \mathbb{E}_x(T_\alpha) - \mathbb{E}_y(T_\beta) + \mathbb{E}_y(T_\alpha)}{\mathbb{E}_\alpha(T_\beta) + \mathbb{E}_\beta(T_\alpha)}$$

By the corollary of reversibility for the circle β, x, α, y

$$\mathbb{E}_\beta(T_x) - \mathbb{E}_\alpha(T_x) - \mathbb{E}_\beta(T_y) + \mathbb{E}_\alpha(T_y) = \mathbb{E}_x(T_\beta) - \mathbb{E}_x(T_\alpha) - \mathbb{E}_y(T_\beta) + \mathbb{E}_y(T_\alpha)$$

Hence $\big(u(x) - u(y)\big)\big(\mathbb{E}_\alpha(T_\beta) + \mathbb{E}_\beta(T_\alpha)\big) = \dfrac{w}{w_{xy}}\, j(x,y)$.

Thompson principle

For any current $j(x,y)$, let $T(j) = \frac{1}{2}\sum_{(x,y)\in E} r_{xy}\big(j(x,y)\big)^2$ be its energy.

The solution of Ohm problem 1 minimizes $\min\{T(j)$ where j is a unit current $\alpha \to \beta\}$.

Dirichlet principle

For any potential $u(x)$, let $D(u) = \frac{1}{2}\sum_{(x,y)\in E} w_{xy}\big(u(x) - u(y)\big)^2$ be its energy.

The solution of Ohm problem 2 minimizes $\min\{D(u)$ where u is a potential with $u(\alpha) - u(\beta) = 1\}$.