Central limit theorems

Theorem (CLT for martingales)

Let $\xi_1, \xi_2, \ldots$ be an ergodic stationary sequence of random variables. Assume that they are the differences of an L^2 martingale, that is $\mathbb{E}(\xi_n \mid \sigma(\xi_1, \ldots, \xi_{n-1})) = 0$. Suppose that $\sigma^2 = \mathbb{E}(\xi_1^2) < \infty$. Then

$$\frac{\xi_1 + \ldots + \xi_n}{\sigma\sqrt{n}} \Rightarrow N(0,1) \quad \text{as } n \to \infty \text{ in distribution.}$$

Theorem (CLT for Markov chains)

Let X_n be ~~an~~ stationary and irreducible Markov chain with stationary distribution π. Let $f \in L^2(S, \pi)$ with $\mathbb{E}_\pi(f) = 0$. Assume that there is a $g \in L^2(S, \pi)$ s.t. $f = (I - P)g$ where P is the transition operator. Then

$$\frac{f(X_1) + \ldots + f(X_n)}{\sigma\sqrt{n}} \Rightarrow N(0,1) \quad \text{as } n \to \infty$$

in distribution where $\sigma^2 = \mathbb{E}(g(X_0)^2) - \mathbb{E}(\mathbb{E}(g(X_1) \mid X_0)^2)$

$= \langle g, g \rangle - \langle Pg, Pg \rangle$

Proof $Pg(x) = \mathbb{E}(g(X_1) \mid X_0 = x)$, hence by $f = (I - P)g$,

$$\sum_{k=1}^{n} f(X_k) = \sum_{k=1}^{n} \left(g(X_k) - \mathbb{E}(g(X_{k+1}) \mid X_k) \right)$$

$$= \sum_{k=1}^{n} \left(g(X_k) - \mathbb{E}(g(X_k) \mid X_{k-1}) \right) + \mathbb{E}(g(X_1) \mid X_0) - \mathbb{E}(g(X_{n+1}) \mid X_n)$$

Then $\xi_k = g(X_k) - \mathbb{E}(g(X_k) \mid X_{k-1})$ forms an ergodic and stationary sequence of L^2 martingale differences, hence the CLT for martingales applies. $\sigma^2 = \mathbb{E}((g(X_1) - \mathbb{E}(g(X_1) \mid X_0))^2) = \mathbb{E}((g(X_1))^2)$
$-2\mathbb{E}(g(X_1)\mathbb{E}(g(X_1) \mid X_0)) + \mathbb{E}((\mathbb{E}(g(X_1) \mid X_0))^2)$

and $\mathbb{E}(g(X_1)\mathbb{E}(g(X_1) \mid X_0)) = \mathbb{E}(\mathbb{E}(g(X_1)\mathbb{E}(g(X_1) \mid X_0) \mid X_0)) = \mathbb{E}((\mathbb{E}(g(X_1) \mid X_0))^2)$.

And $\frac{\mathbb{E}(g(X_1) \mid X_0)}{\sqrt{n}}$, $\frac{\mathbb{E}(g(X_{n+1}) \mid X_n)}{\sqrt{n}} \to 0$ in $\mathbb{P}$.

Example

$S = \{1,2\}$ $\qquad P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \Rightarrow \pi = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\mathbb{E}_\pi(f) = 0 \qquad I - P = \begin{pmatrix} (1-p) & -(1-p) \\ -(1-p) & (1-p) \end{pmatrix}.$

We solve $(I-P)g = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow g\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-p} + a \\ a \end{pmatrix} \quad a \in \mathbb{R}$

We chose $a = 0$, $g = \begin{pmatrix} \frac{1}{1-p} \\ 0 \end{pmatrix}$

$$\mathbb{E}(g(X_1) \mid X_0) = \begin{cases} \frac{p}{1-p} & \text{if } X_0 = 1 \\ 1 & \text{if } X_0 = 2 \end{cases}$$

$$\sigma^2 = \mathbb{E}(g(X_0)^2) - \mathbb{E}((\mathbb{E} g(X_1) \mid X_0))^2) = \frac{1}{2}\frac{1}{(1-p)^2} - \left(\frac{1}{2}\left(\frac{p}{1-p}\right)^2 + \frac{1}{2}\right)$$

$$= \frac{1 - p^2 - (1-p)^2}{2(1-p)^2} = \frac{2p - 2p^2}{2(1-p)^2} = \frac{p}{1-p}$$

Remark

- $p \to 0 \qquad \sigma \to 0$
- $p \to 1 \qquad \sigma \to \infty$