Stationary processes

Def

$(\Omega, \mathcal{F}, \mathbb{P})$, $X_0, X_1, X_2, \ldots : \Omega \to S$ random variables, i.e. measurable functions where $(S, \mathcal{S})$ is a measurable space.

They are *stationary* if for every k and n

$$(X_0, X_1, \ldots, X_n) \stackrel{d}{=} (X_k, X_{k+1}, \ldots, X_{k+n}).$$

Examples

- $X_0, X_1, X_2, \ldots$ iid.
- stationary MC, state space S discrete, P_{xy} stoch. matrix
  π stat. distr. on S, i.e. $\sum_{x \in S} \pi(x) P_{xy} = \pi(y)$.
  $\mathbb{P}(X_0 = x) = \pi(x)$ and $X_0, X_1, X_2, \ldots$ MC with transition matrix P.
  Then $\mathbb{P}(X_k = x_k, X_{k+1} = x_{k+1}, \ldots, X_{k+n} = x_{k+n}) = \pi(x_k) P_{x_k x_{k+1}} \cdots P_{x_{k+n-1} x_{k+n}}$.
- rotation of the circle
  $\Omega = [0,1)$, $\mathcal{F} =$ Borel, $P =$ Lebesgue
  $\theta \in (0,1)$ fixed, $X_n(\omega) = \{\omega + n\theta\}$
- Bernoulli shift
  $\Omega = [0,1)$, $\mathcal{F} =$ Borel, $\mathbb{P} =$ Lebesgue $\qquad X_n(\omega) = \{2^n \omega\}$

Measure preserving transformations

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad T : \Omega \to \Omega$ measurable s.t. $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for any $A \in \mathcal{F}$, then T is measure preserving.

$(\Omega, \mathcal{F}, \mathbb{P}, T)$ is called a dynamical system

If $g : \Omega \to S$ is measurable/random variable, then $X_n(\omega) = g(T^n \omega)$ defines a stationary sequence

Kolmogorov's extension thm

Suppose that the prob measures μ_n on $(\mathbb{R}^n, \mathcal{S}^n)$ are consistent in the sense

$$\mu_{n+1}((a_1,b_1] \times \dots \times (a_n,b_n] \times S) = \mu_n((a_1,b_1] \times \dots \times (a_n,b_n])$$

Then $\exists!$ P prob. measure on $(S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}})$ with

$$P(\omega : \omega_i \in (a_i,b_i] \text{ for } i=1,\dots,n) = \mu_n((a_1,b_1] \times \dots \times (a_n,b_n]).$$

Prop

Every stationary sequence can be ~~obtained~~ by a ~~measure~~ preserving transformation.

Proof: Let $Y_0, Y_1, Y_2, \dots$ be stationary. By Kolmogorov's extension, $\exists$ P on $(S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}})$ s.t. $X_n(\omega) = \omega_n$ has the same distr. as Y_n.
Let T be the shift, i.e. $T(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$
and $g(\omega) = \omega_0$. Then T is measure preserving and
$X_n(\omega) = g(T^n \omega)$.

Ergodicity

$(\Omega, \mathcal{F}, \mathbb{P}, T)$ dyn. sys. $A \in \mathcal{F}$ is invariant if $T^{-1}A = A$ a.s.
$\mathcal{I} = \{A \in \mathcal{F} \text{ invariant}\}$ σ-alg.
The dyn. sys is ergodic if $\mathcal{I}$ is trivial, i.e. $A \in \mathcal{I} \Rightarrow \mathbb{P}(A) = 0$ or 1.

Examples

- iid. $A \in \mathcal{I}$: $\Omega = S^{\mathbb{N}}$ can be chosen $\{\omega : \omega \in A\} = \{\omega : T\omega \in A\}$ a.s.
  $\Rightarrow A \in \sigma(X_1, X_2, \dots)$ by iteration $A \in \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots) = \mathcal{T} \Rightarrow \mathcal{I} \subset \mathcal{T}$
  By Kolmogorov's 0-1 law $\mathcal{T}$ is trivial $\Rightarrow \mathcal{I}$ is trivial.
- discrete stationary Markov chains : state space S, $\Omega = S^{\mathbb{N}}$, $\mathbb{P}$, T, stat. distr. π
  ergodicity $\Leftrightarrow$ irreducibility
  $\boxed{\Rightarrow}$: if it were not irreducible, $S = S_1 \cup S_2$ $A = \{\omega : \omega_0 \in S_1\} \in \mathcal{I}$ non-triv.

$\boxed{\Leftarrow}$ $A \in \mathcal{I}$ $\quad \mathcal{F}_n = \sigma(X_0, \dots, X_n)$ also for $\mathcal{F}_\infty$

$\mathbb{1}_A = \mathbb{1}_A(T^n \cdot)$ by invariance $\Rightarrow \mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n) = \mathbb{E}(\mathbb{1}_A(T^n \cdot) \mid \mathcal{F}_n) = h(X_n)$

On the other hand, by Lévy's upward thm. $\mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_n) \to \mathbb{E}(\mathbb{1}_A \mid \mathcal{F}_\infty) = \mathbb{1}_A$ a.s.

Hence $h(X_n) \to \mathbb{1}_A$ a.s. as $n \to \infty$. Since X_n is irreducible, it visits all points infinitely often a.s., the only possibility is $h \equiv 0$ or $h \equiv 1$ and $\mathbb{P}(A) = 0$ or 1.

- rotation of the circle

  ergodicity $\Leftrightarrow$ θ is irrational

  $\boxed{\Rightarrow}$ $\theta = \frac{m}{n}$ with $m < n$ pos. int. and B is Borel subset of $[0, \frac{1}{n})$,

  then $\bigcup_{k=0}^{n-1} (B + \frac{k}{n})$ is an invariant set

  $\boxed{\Leftarrow}$ $f \in L^2([0,1), \mathcal{F}, \mathbb{P})$ $\qquad f(\omega) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \omega}$

  with $c_k = \int e^{-2\pi i k \omega} f(\omega)\, d\omega$ Fourier series (unique)

  $f(T\omega) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k(\omega + \theta)} = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \theta} e^{2\pi i k \omega}$

  f is invariant $\Leftrightarrow$ $c_k(e^{2\pi i k \theta} - 1) = 0$, i.e. $c_k = 0$ for $k \neq 0$,

  hence f is constant. For $f = \mathbb{1}_A$ with $A \in \mathcal{I}$ $\Rightarrow A = \emptyset$ or $A = \Omega$.

- Bernoulli shift is ergodic

  $$X_n = \sum_{m=0}^{\infty} 2^{-(m+1)} Y_{n+m} \quad \text{where } Y_0, Y_1, Y_2, \dots \text{ are iid } \mathrm{Ber}(\tfrac{1}{2})$$

Ergodic theorems

Theorem (Neumann, L^2 ergodic thm)

Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be a dynamical system, i.e. $T: \Omega \to \Omega$ measure preserving. Let $\mathcal{I}$ be the invariant σ-algebra. Then for any $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$,

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k \cdot) \longrightarrow \mathbb{E}(f \mid \mathcal{I}) \quad \text{in } L^2 \text{ as } n \to \infty.$$

Remark

If T is ergodic, $\mathcal{I}$ is trivial, $\mathbb{E}(f \mid \mathcal{I}) = \mathbb{E}f$.

Proof Let $(Uf)(\omega) = f(T\omega)$ define an isometry in $L^2(\Omega,\mathcal{F},\mathbb{P})$.

Indeed, $\langle Uf, Ug\rangle = \mathbb{E}\left(f(T\omega)\cdot g(T\omega)\right) \underset{\substack{\uparrow \\ T \text{ measure preserving}}}{=} \mathbb{E}\left(f(\omega)\,g(\omega)\right) = \langle f, g\rangle$

$\int_\Omega f(T\omega)g(T\omega)\,d\mathbb{P}(\omega) = \int_\Omega f(\omega)g(\omega)\,d\mathbb{P}(\omega)$

$\mathcal{K} = \{ f \in L^2, \mathcal{I} \text{ measurable}\} \subseteq L^2(\Omega,\mathcal{F},\mathbb{P})$ closed subspace

$\mathcal{K} = \mathrm{Ker}(U - I)$. ~~Define Π to be the orthogonal projection to $\mathcal{K}$.~~

~~Then $\Pi f = \mathbb{E}(f \mid \mathcal{I})$~~ The orthogonal projection ~~of~~ f to $\mathcal{K}$ is $\mathbb{E}(f\mid\mathcal{I})$.

Fact

If $\mathcal{C} \subseteq \mathcal{H}$ is a closed subspace of a Hilbert space, then $\mathcal{H} = \mathcal{C} \oplus \mathcal{C}^\perp$ where $\mathcal{C}^\perp = \{h \in \mathcal{H} : \langle h, g\rangle = 0 \ \forall g \in \mathcal{C}\}$ in the sense that any $h \in \mathcal{H}$ is written uniquely as $h = c + c'$ with $c \in \mathcal{C}$ and $c' \in \mathcal{C}^\perp$.

Proof (fact): by the projection thm, $\exists!\, c \in \mathcal{C}$ with $\langle h - c, w\rangle = 0 \ \forall w \in \mathcal{C}$.

Next we show that $\mathcal{K} = \overline{\mathrm{Ran}(U-I)}^\perp$ where $\mathcal{K} = \mathrm{Ker}(U - I)$

$\boxed{\subseteq}$: $f \in \mathrm{Ker}(U-I)$ and $Ug - g \in \mathrm{Ran}(U-I) \Rightarrow \langle f, Ug - g\rangle = \langle \underbrace{(U-I)f}_{0}, g\rangle = 0$

$\boxed{\supseteq}$: $h \in \overline{\mathrm{Ran}(U-I)}^\perp$: $\langle h, (U-I)h\rangle = 0$. We show that $(U-I)h = 0$

$\langle (U-I)h, (U-I)h\rangle = \langle Uh, (U-I)h\rangle - \langle \underbrace{h, (U-I)h}_{0}\rangle = \langle Uh, Uh\rangle - \langle Uh, h\rangle =$

$= \langle h, h\rangle - \langle Uh, h\rangle = \langle (I-U)h, h\rangle = 0$

Let $f \in L^2(\Omega,\mathcal{F},\mathbb{P})$, if $f = Ug - g$, then

$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\cdot) = \underbrace{\frac{1}{n}(U^n g - g)}_{\downarrow 0 \text{ in } L^2}$ and $\overset{f}{h} \in \mathcal{K}$ $\frac{1}{n}\sum_{k=0}^{n-1} h(T^k\cdot) = h = \mathbb{E}(h\mid\mathcal{I})$

Theorem (Birkhoff, L^1 ergodic thm) 1931

Let $f \in L^1(\Omega,\mathcal{F},\mathbb{P})$ and $T: \Omega \to \Omega$ measure preserving. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\cdot) \longrightarrow \mathbb{E}(f\mid\mathcal{I}) \quad \text{a.s. and in } L^1 \text{ as } n\to\infty.$$

Proof by Garsia 1965

Lemma (maximal ergodic)

$X_j(\omega) = f(T^j\omega)$, $S_k = X_0 + X_1 + \dots + X_{k-1}$ and $M_k = \max(0, S_1, \dots, S_k)$.

Then $\mathbb{E}(X_0 \cdot \mathbb{1}_{\{M_k > 0\}}) \geq 0$.

Proof (lemma)

for $j \le k$ $\quad S_j(T\omega) \le M_k(T\omega)$

$$S_{j+1}(\omega) = X_0(\omega) + S_j(T\omega) \le X_0(\omega) + M_k(T\omega)$$

$\Rightarrow X_0(\omega) \ge S_{j+1}(\omega) - M_k(T\omega)$ for $j = 1, \dots, k$

Also $X_0(\omega) \ge S_1(\omega) - M_k(T\omega)$ since $S_1(\omega) = X_0(\omega)$ and $M_k(T\omega) \ge 0$.

Therefore $\mathbb{E}(X_0 \cdot \mathbb{1}_{\{M_k > 0\}}) \ge \mathbb{E}\left(\left(\max(S_1, \dots, S_k) - M_k(T\omega)\right) \mathbb{1}_{\{M_k > 0\}}\right)$

$$= \mathbb{E}\left(\left(M_k(\omega) - M_k(T\omega)\right) \mathbb{1}_{\{M_k > 0\}}\right)$$

$$\ge \mathbb{E}\left((M_k(\omega) - M_k(T\omega)) \mathbb{1}_{\{M_k > 0\}}\right) + \mathbb{E}\left(\underbrace{(M_k(\omega) - M_k(T\omega))}_{\le 0} \mathbb{1}_{\{M_k = 0\}}\right)$$

$$= \mathbb{E}\left(M_k(\omega) - M_k(T\omega)\right) = 0$$ since T is measure preserving.

Proof (thm) by subtracting $\mathbb{E}(f \mid \mathcal{I})$, we may assume that $\mathbb{E}(f \mid \mathcal{I}) = 0$. Let $\bar{X} = \limsup \frac{S_n}{n}$ and let $\varepsilon > 0$.

Define $D = \{\bar{X} > \varepsilon\} \in \mathcal{I}$, the goal is $\mathbb{P}(D) = 0$.

$X^* = (f - \varepsilon)\mathbb{1}_D \qquad S_n^* = X^*(\omega) + X^*(T\omega) + \dots + X^*(T^{n-1}\omega)$

$M_n^* = \max(0, S_1^*, \dots, S_n^*) \qquad F_n = \{M_n^* > 0\} \qquad F = \bigcup_n F_n = \{\sup_{k \ge 1} \frac{S_k^*}{k} > 0\}$

Since $X^* = (f - \varepsilon)\mathbb{1}_D$, $\quad F = \{\sup_{k \ge 1} \frac{S_k}{k} > \varepsilon\} \cap D = D$

By the lemma $\mathbb{E}(X^* \mathbb{1}_{F_n}) \ge 0$

$\downarrow$ dom. conv.

$$\mathbb{E}(X^* \cdot \mathbb{1}_F) = \mathbb{E}(X^* \mathbb{1}_D) = \mathbb{E}((f - \varepsilon)\mathbb{1}_D) = \mathbb{E}\big((\overbrace{\mathbb{E}(f \mid \mathcal{I})}^{=0} - \varepsilon)\mathbb{1}_D\big) = -\varepsilon \mathbb{P}(D)$$

$\Rightarrow \mathbb{P}(D) = 0$, since $S_n/n \to 0$ a.s.

For the L^1 conv, let $f_M = f \mathbb{1}_{\{|f| \le M\}}$

By the previous part, $\frac{1}{n}\sum_{k=0}^{n-1} f_M(T^k \cdot) \to \mathbb{E}(f_M \mid \mathcal{I})$ a.s. and by the bounded conv thm, it also conv. in L^1.

On

$$\mathbb{E}\left|\frac{1}{n}\sum_{k=0}^{n-1}(f - f_M)(T^k \cdot) - \mathbb{E}(f - f_M \mid \mathcal{I})\right| \underset{\substack{\uparrow \\ \text{triangle ineq.} \\ + \text{Jensen}}}{\le} 2\,\mathbb{E}|f - f_M| \to 0 \text{ as } M \to \infty$$

Consequences

- For iid sequences, strong LLN
- stationary MC: X_n irred. MC and $f : S \to \mathbb{R}$ with

$$\sum_{x\in S} |f(x)|\,\pi(x) < \infty \text{, then } \frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \longrightarrow \sum_{x\in S} f(x)\pi(x) \text{ a.s. and in } L^1$$

- rotation of the circle with $\theta \in (0,1)$ irrational, then

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{\{\omega + k\theta\} \in [a,b)\}} \to b-a \quad \text{a.s. } \omega \in [0,1)$$

Thm (Weyl's equidistribution thm)

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{\{\omega+k\theta\}\in[a,b)\}} \to b-a \quad \text{for all } \omega\in[0,1).$$

Proof $A_j = [a+\frac{1}{j}, b-\frac{1}{j})$ and assume $b-a > \frac{2}{j}$

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k\omega \in A_j\}} \to b-a-\frac{2}{j} \quad \text{for all } \omega\in\Omega_j \text{ with } \mathbb{P}(\Omega_j)=1$$

$G = \bigcap_{j=1}^{\infty} \Omega_j$ has $\mathbb{P}(G)=1$ and it is dense.

If $x\in[0,1)$, $\exists\, \omega_j \in G$ s.t. $|x-\omega_j| < \frac{1}{j}$ and $T^k\omega_j \in A_j \Rightarrow T^k x \in [a,b)$

Hence $\liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x\in[a,b)\}} \geq b-a-\frac{2}{j}$

Consequence

$$\frac{\#\{k : 1\leq k\leq n : \text{first digit of } 2^k \text{ is } m\}}{n} = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{\{0+k\cdot\log_{10}2\}\in[\log_{10}m,\log_{10}(m+1))\}}$$

$$\to \log_{10}(m+1) - \log_{10} m \quad \text{as } n\to\infty \qquad m=1,2,\dots,9 \quad \text{Benford's law}$$

- Bernoulli shift: for any $i_1,\dots,i_k \in\{0,1\}$, the density of the string $i_1 i_2 \dots i_k$ in the binary expansion of almost every ω is 2^{-k}.