

Exercises in Extreme value theory

2024 spring semester

1. **Homework 1.A (26th Feb 2024)** Show that $L(t) = \log t$ is a slowly varying function but t^ϵ is not if $\epsilon \neq 0$.
2. If the random variable X has distribution F with finite variance, then show that the condition

$$\lim_{x \rightarrow \infty} \frac{x^2 \mathbf{P}(|X| > x)}{\int_{|y| \leq x} y^2 dF(y)} = 0$$

automatically holds.

Hint: Prove and use the inequality

$$x^2 \mathbf{P}(|X| > x) \leq \int_{|y| \geq x} y^2 dF(y).$$

3. **Homework 1.B (26th Feb 2024)** Let W_1 and W_2 be independent standard normal random variables.
 - (a) Check that $1/W_2^2$ has the Lévy distribution, that is, the stable law with index $\alpha = 1/2$ and $\kappa = 1$, i.e. it has density

$$\frac{1}{\sqrt{2\pi y^3}} \exp\left(-\frac{1}{2y}\right) \quad \text{for } y > 0.$$

(b) Prove that W_1/W_2 has Cauchy distribution.

4. Let $\tau = \inf\{t > 0 : B_t = 1\}$ be the hitting time of level 1 by the standard Brownian motion B_t which starts from $B_0 = 0$. Show that τ has the Lévy distribution, that is, its density is $(2\pi y^3)^{-1/2} \exp(-1/(2y))$ for $y > 0$.

Hint: It can be used without proof that as a consequence of the reflection principle

$$\mathbf{P}(\max\{B_s : s \in [0, t]\} > x) = 2 \left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

where $\Phi(x) = \int_{-\infty}^x (2\pi)^{-1/2} \exp(-y^2/2) dy$ is the standard normal distribution function. The distribution function of τ can be directly expressed by the probability in the equality above.

5. **Homework 1.C (26th Feb 2024)** Let X have a symmetric stable law with index α . Show that $\mathbf{E}|X|^p < \infty$ for $p \in (0, \alpha)$.

Hint: If $\varphi(t) = \mathbf{E}(e^{itX})$ denotes the characteristic function of X , then the following inequality can be used without proof for $u > 0$:

$$\mathbf{P}\left(|X| > \frac{2}{u}\right) \leq \frac{1}{u} \int_{-u}^u (1 - \varphi(t)) dt.$$

Further, the identity

$$\mathbf{E}(|X|^p) = p \int_0^\infty t^{p-1} \mathbf{P}(|X| > t) dt$$

holds.

6. **Homework 2.A (11th Mar 2024)** Let X_1, X_2, \dots be an iid. sequence of Poisson random variables with parameter $\lambda > 0$, that is,

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \dots$ and let $M_n := \max_{1 \leq i \leq n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converge to 0 for the integers $k \rightarrow \infty$.

7. Let X_1, X_2, \dots be an iid. sequence of negative binomial random variables with parameters $p \in (0, 1)$ and $m \in \mathbb{N}$, that is,

$$\mathbf{P}(X = k) = \binom{k+m-1}{k} p^m (1-p)^k$$

for $k = 0, 1, 2, \dots$ and let $M_n := \max_{1 \leq i \leq n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converges to $1-p$ for the integers $k \rightarrow \infty$. To this end, show and use the asymptotic identity

$$\mathbf{P}(X = k) = \frac{k^{m-1}}{(m-1)!} p^m (1-p)^k (1 + o(1))$$

as $k \rightarrow \infty$.

8. **Homework 2.B (11th Mar 2024)** Suppose that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies Cauchy's functional equation

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Prove that for any $z \in \mathbb{R}$ the value of the function $f(z)$ determines all the values $f\left(\frac{p}{q}z\right)$ for any $\frac{p}{q} \in \mathbb{Q}$ rational.

9. For $f : \mathbb{R} \rightarrow \mathbb{R}$ functions, consider Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

Prove that the only solutions for the equation are the linear functions $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$, if we assume that f is

- (a) continuous;
- (b) continuous at one point;
- (c) monotonic on any interval;
- (d) bounded on any interval;
- (e) **Homework 2.C (11th Mar 2024)** measurable.

Hint: Lusin's theorem ensures that by the measurability of f there is a subset $F \subseteq [0, 1]$ with Lebesgue measure $2/3$ such that f is uniformly continuous on F . Then for any $h \in (0, 1/3)$, the sets F and $F - h = \{x - h : x \in F\}$ cannot be disjoint. Use this to conclude that the function f is bounded on an interval $(0, \delta)$ for some small $\delta > 0$.

10. Let X_1, X_2, \dots be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which have the same distribution function $F(x)$, but we do not assume anything about their dependence structure. Let $M_n := \max_{1 \leq i \leq n} |X_i|$.

- (a) **Homework 3.A (25th Mar 2024)** Suppose that there is an $\alpha > 0$ such that $\int_{-\infty}^{\infty} |x|^\alpha dF(x) < \infty$, that is, $\mathbf{E}(|X_i|^\alpha) < \infty$. Prove that for any $\varepsilon > 0$ and for fixed $\delta > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}(n^{-(1/\alpha + \varepsilon)} |M_n| > \delta) = 0.$$

- (b) Suppose that there is an $s > 0$ such that $\int_{-\infty}^{\infty} e^{s|x|} dF(x) < \infty$, that is, $\mathbf{E}(e^{s|X_i|}) < \infty$. Prove that for any sequence b_n that goes to ∞ and for fixed $\delta > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}((b_n \log n)^{-1} |M_n| > \delta) = 0.$$

Hint: Use the following inequality

$$\mathbf{P}\left(\max_{1 \leq i \leq n} |X_i| > \lambda\right) = \mathbf{P}(\cup_{i=1}^n \{|X_i| > \lambda\}) \leq \sum_{i=1}^n \mathbf{P}(|X_i| > \lambda) = n\mathbf{P}(|X_1| > \lambda)$$

and a Markov type inequality.

11. Let X_1, X_2, \dots be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \leq i \leq n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \rightarrow \infty} x^\alpha (1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim bx^{-\alpha}$ as $x \rightarrow \infty$), then show that the distribution of $(bn)^{-1/\alpha} M_n$ converges weakly to the Fréchet distribution:

$$\mathbf{P}((bn)^{-1/\alpha} M_n < x) \rightarrow \mathbf{1}(x > 0) \exp(-x^{-\alpha}).$$

12. **Homework 3.B (25th Mar 2024)** Let X_1, X_2, \dots be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \leq i \leq n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \rightarrow \infty} e^{\lambda x} (1 - F(x)) = b$ for some fixed constants $\lambda, b \in (0, \infty)$ (that is, $1 - F(x) \sim be^{-\lambda x}$ as $x \rightarrow \infty$), then show that the distribution of $\lambda M_n - \log(bn)$ converges weakly to the Gumbel distribution:

$$\mathbf{P}(\lambda M_n - \log(bn) < x) \rightarrow \exp(-e^{-x}).$$

13. Let X_1, X_2, \dots be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \leq i \leq n} X_i$. If $F(x_0) = 1$ and $F(x) < 1$ for all $x < x_0$ and $\lim_{x \rightarrow x_0} (x_0 - x)^{-\alpha} (1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim b(x_0 - x)^\alpha$ as $x \rightarrow x_0$), then show that the distribution of $(bn)^{1/\alpha} (M_n - x_0)$ converges weakly to the Weibull distribution:

$$\mathbf{P} \left((bn)^{1/\alpha} (M_n - x_0) < x \right) \rightarrow \mathbb{1}(x < 0) \exp(-(-x)^\alpha) + \mathbb{1}(x \geq 0).$$

14. Let X_1, X_2, \dots be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Suppose that the common distribution has a finite right endpoint, that is,

$$x_F = \sup\{x \in \mathbb{R} : F(x) < 1\} < \infty$$

and that the right endpoint has a positive mass $\mathbf{P}(X_i = x_F) = \overline{F}(x_F) > 0$. Let $M_n := \max_{1 \leq i \leq n} X_i$. Prove that for any sequence of reals (u_n) , if the sequence of probabilities $\mathbf{P}(M_n < u_n)$ converges, then the limit is either 0 or 1.

Hint: Use the proposition about Poisson approximation to see that it is enough to show that $n\overline{F}(u_n)$ goes either to 0 or to ∞ .

15. Let X be a random variable with distribution function F where the tail is $1 - F(x) = x^{-\alpha}$ for $x \geq 1$ with some $\alpha > 0$. Then we know that $F \in \text{MDA}(\Phi_\alpha)$. Which MDA does the distribution of X^p and that of $\ln(X)$ belong to if $p > 0$? What are the normalization constants?
16. Suppose that $X > 0$ is a random variable and $\alpha > 0$ is a number. Show that the following are equivalent:

- (a) X has the Fréchet distribution function

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$$

- (b) $\ln X^\alpha$ has the Gumbel distribution function

$$\Lambda(x) = \exp(-e^{-x})$$

- (c) $-X^{-1}$ has the Weibull distribution function

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

17. **Homework 3.C (25th Mar 2024)** Let X_1, X_2, \dots be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k . Denote by $m_n = \min(X_1, \dots, X_n)$ the minimum record up to n and let $m_\infty = \lim_{n \rightarrow \infty} m_n$ which exists since m_n is non-increasing.

- (a) Show that if $\sum_{k=1}^{\infty} \lambda_k < \infty$, then with probability one, the minimum record is broken finitely many times and $m_\infty > 0$.
- (b) Show that if $\sum_{k=1}^{\infty} \lambda_k = \infty$, then with probability one, the minimum record is broken infinitely many times and $m_\infty = 0$.

Hint: One can use the representation of X_k as the first point of a Poisson point process on \mathbb{R}_+ with intensity λ_k and the fact that the union of independent Poisson point processes with intensities $\lambda_1, \dots, \lambda_k$ is a Poisson point process with intensity $\lambda_1 + \dots + \lambda_k$.

18. Let X_1, X_2, \dots be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k . Denote by $M_n = \max(X_1, \dots, X_n)$ the maximum record up to n and let $M_\infty = \lim_{n \rightarrow \infty} M_n$ which exists since M_n is non-decreasing.

- (a) Show that for any constant $K > 0$ if

$$\sum_{n=1}^{\infty} e^{-\lambda_n K} = \infty,$$

then with probability one there are infinitely many indices n such that $X_n > K$. Further, for any $K > 0$ if

$$\sum_{n=1}^{\infty} e^{-\lambda_n K} < \infty,$$

then with probability one there are finitely many indices n such that $X_n > K$.

Hint: Use the Borel–Cantelli lemmas.

- (b) Conclude that if $\sum_{n=1}^{\infty} e^{-\lambda_n K} = \infty$ for any $K > 0$, then with probability one, $M_\infty = \infty$ and the maximum record is broken infinitely many times. If $\sum_{n=1}^{\infty} e^{-\lambda_n K} < \infty$ for any $K > 0$, then with probability one, $M_\infty < \infty$ and the maximum record is broken finitely many times.
- (c) If there are $K_1 > K_2 > 0$ such that $\sum_{n=1}^{\infty} e^{-\lambda_n K_1} < \infty$ but $\sum_{n=1}^{\infty} e^{-\lambda_n K_2} = \infty$, then there is a critical K_c such that for smaller values of K the sum is infinite and for larger values of K the sum is finite. Show that in this case with probability one

$$\limsup_{n \rightarrow \infty} X_n = K_c.$$

- (d) Suppose that there is a critical K_c as described above. Show that if

$$\sum_{n=1}^{\infty} e^{-\lambda_n K_c} = \infty,$$

then with probability one the maximum record is broken finitely many times.

Hint: The condition shows that there are infinitely many indices n such that $X_n > K_c$, but for any $\varepsilon > 0$, there are only finitely many with $X_n > K_c + \varepsilon$.

- (e) Suppose that there is a critical K_c as described above. Show that if

$$\sum_{n=1}^{\infty} e^{-\lambda_n K_c} < \infty,$$

then with positive probability the maximum record is broken infinitely many times. Compute this probability.

Hint: Observe that the maximum record is broken infinitely many times if and only if $X_n < K_c$ for all indices n , since $\limsup_{n \rightarrow \infty} X_n = K_c$.