Extreme value theory midterm exam, 4 May 2022

1. (a) What does it mean that a function is slowly varying at 0?

   (b) Which of the functions

   $$f(x) = \sin(\sqrt{x}), \qquad g(x) = \sin\left(\frac{1}{\log x}\right)$$

   defined for $x > 0$ are slowly varying at 0?

2. For functions $f : \mathbb{R} \to \mathbb{R}$, consider Cauchy's functional equation

   $$f(x + y) = f(x) + f(y).$$

   Prove that the only solutions for the equation are the linear functions $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$ if we assume that f is monotonic on $\mathbb{R}$.

3. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x\to\infty} x^\alpha(1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim bx^{-\alpha}$ as $x \to \infty$), then show that the distribution of $(bn)^{-1/\alpha} M_n$ converges weakly to the Fréchet distribution:

   $$\mathbf{P}\left((bn)^{-1/\alpha} M_n < x\right) \to \mathbb{1}(x > 0) \exp\left(-x^{-\alpha}\right).$$

4. Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed uniform random variables on $[0, 1]$. Let $N_n = \min(U_1^2, \ldots, U_n^2)$ for $n = 1, 2, \ldots$ Which non-trivial limit distribution does the renormalized sequence of N_n converge to? Under what normalization?

   Hint: Use the fact that $\min(U_1^2, \ldots, U_n^2) = -\max(-U_1^2, \ldots, -U_n^2)$ and compute the tail probability function of the random variable $-U_1^2$.

5. Consider the tail probability function

   $$\overline{F}(x) = \begin{cases} e^{-\frac{2}{1-x}} & \text{if } x < 1, \\ 0 & \text{if } x \ge 1. \end{cases}$$

   Compute the corresponding hazard rate function $h(x)$. Check the condition to belong to the maximum domain of attraction of the Gumbel distribution, that is,

   $$\frac{\overline{F}(x + t/h(x))}{\overline{F}(x)} \to e^{-t}$$

   for all $t \in \mathbb{R}$ as x goes to the right endpoint of the distribution. Compute the normalization constants $b_n = \overline{F}^{-1}(1/n)$ and $a_n = 1/h(b_n)$ as well.

6. Let $X_2, X_3, \ldots$ be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter $\lambda_k = \log k + 2 \log \log k$. Denote by $M_n = \max(X_2, \ldots, X_n)$ the maximum record up to n.

   (a) Show that for all $K \in (0, 1)$,

   $$\sum_{n=2}^{\infty} e^{-\lambda_n K} = \infty,$$

   and for all $K \geq 1$,

   $$\sum_{n=2}^{\infty} e^{-\lambda_n K} < \infty$$

   holds.

   (b) Conclude that with probability one

   $$\limsup_{n \to \infty} X_n = 1.$$

   Hint: It can be used *without proof* that with probability one for all $K \in (0, 1)$ there are infinitely many indices n such that $X_n > K$ whereas for all $K \geq 1$ there are finitely many indices n such that $X_n > K$.

   (c) Show that with positive probability the maximum record is broken infinitely many times. Compute this probability.

   Hint: Observe that the maximum record is broken infinitely many times if and only if $X_n < 1$ for all indices n, since $\limsup_{n \to \infty} X_n = 1$.