Exercises in Extreme value theory

2022 spring semester

1. Show that $L(t) = \log t$ is a slowly varying function but t^ϵ is not if $\epsilon \neq 0$.

2. **Homework 1.A (2nd Mar 2022)** If the random variable X has distribution F with finite variance, then show that the condition

$$\lim_{x\to\infty} \frac{x^2 \mathbf{P}(|X| > x)}{\int_{|y|\le x} y^2 \mathrm{d}F(y)} = 0$$

automatically holds.

Hint: Prove and use the inequality

$$x^2 \mathbf{P}(|X| > x) \le \int_{|y|\ge x} y^2 \mathrm{d}F(y).$$

3. Let W_1 and W_2 be independent standard normal random variables.

   (a) Check that $1/W_2^2$ has the Lévy distribution, that is, the stable law with index $\alpha = 1/2$ and $\kappa = 1$, i.e. it has density

$$\frac{1}{\sqrt{2\pi y^3}} \exp\left(-\frac{1}{2y}\right) \quad \text{for } y > 0.$$

   (b) Prove that W_1/W_2 has Cauchy distribution.

4. **Homework 1.B (2nd Mar 2022)** Let $\tau = \inf\{t > 0 : B_t = 1\}$ be the hitting time of level 1 by the standard Brownian motion B_t which starts from $B_0 = 0$. Show that τ has the Lévy distribution, that is, its density is $(2\pi y^3)^{-1/2} \exp(-1/(2y))$ for $y > 0$.

Hint: It can be used without proof that as a consequence of the reflection principle

$$\mathbf{P}\left(\max\{B_s : s \in [0,t]\} > x\right) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

where $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2)\,\mathrm{d}y$ is the standard normal distribution function. The distribution function of τ can be directly expressed by the probability in the equality above.

5. **Homework 1.C (2nd Mar 2022)** Let X have a symmetric stable law with index α. Show that $\mathbf{E}|X|^p < \infty$ for $p \in (0, \alpha)$.

Hint: If $\varphi(t) = \mathbf{E}(e^{itX})$ denotes the characteristic function of X, then the following inequality can be used without proof for $u > 0$:

$$\mathbf{P}\left(|X| > \frac{2}{u}\right) \le \frac{1}{u}\int_{-u}^{u} (1 - \varphi(t))\,\mathrm{d}t.$$

Further, the identity

$$\mathbf{E}(|X|^p) = p \int_0^\infty t^{p-1} \mathbf{P}(|X| > t)\, \mathrm{d}t$$

holds.

6. Let $X_1, X_2, \ldots$ be an iid. sequence of Poisson random variables with parameter $\lambda > 0$, that is,

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \ldots$ and let $M_n := \max_{1 \le i \le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converge to 0 for the integers $k \to \infty$.

7. **Homework 2.A (23rd Mar 2022)** Let $X_1, X_2, \ldots$ be an iid. sequence of negative binomial random variables with parameters $p \in (0,1)$ and $m \in \mathbb{N}$, that is,

$$\mathbf{P}(X = k) = \binom{k+m-1}{k} p^m (1-p)^k$$

for $k = 0, 1, 2, \ldots$ and let $M_n := \max_{1 \le i \le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converges to $1-p$ for the integers $k \to \infty$. To this end, show and use the asymptotic identity

$$\mathbf{P}(X = k) = \frac{k^{m-1}}{(m-1)!} p^m (1-p)^k (1 + o(1))$$

as $k \to \infty$.

8. **Homework 2.B (23rd Mar 2022)** Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Cauchy's functional equation

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Prove that the value of the function $f(z)$ at any $z \in \mathbb{R}$ determines all the values $f\left(\frac{p}{q} z\right)$ for any $\frac{p}{q} \in \mathbb{Q}$ rational.

9. For $f : \mathbb{R} \to \mathbb{R}$ functions, consider Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

Prove that the only solutions for the equation are the linear functions $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$, if we assume that f is

(a) continuous;

(b) continuous at one point;

(c) monotonic on any interval;

(d) **Homework 2.C (23rd Mar 2022)** bounded on any interval;

(e) measurable.

Hint: Lusin's theorem ensures that by the measurability of f there is a subset $F \subseteq [0,1]$ with Lebesgue measure $2/3$ such that f is uniformly continuous on F. Then for any $h \in (0, 1/3)$, the sets F and $F - h = \{x - h : x \in F\}$ cannot be disjoint. Use this to conclude that the function f is bounded on an interval $(0, \delta)$ for some small $\delta > 0$.

10. Let $X_1, X_2, \dots$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which have the same distribution function $F(x)$, but we do not assume anything about their dependence structure. Let $M_n := \max_{1 \le i \le n} |X_i|$.

(a) Suppose that there is an $\alpha > 0$ such that $\int_{-\infty}^{\infty} |x|^\alpha \mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(|X_i|^\alpha) < \infty$. Prove that for any $\varepsilon > 0$ and for fixed $\delta > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left(n^{-(1/\alpha + \varepsilon)} |M_n| > \delta\right) = 0.$$

(b) Suppose that there is an $s > 0$ such that $\int_{-\infty}^{\infty} \exp(s|x|) \mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(\exp(s|X_i|)) < \infty$. Prove that for any sequence b_n that goes to ∞ and for fixed $\delta > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left((b_n \log n)^{-1} |M_n| > \delta\right) = 0.$$

Hint: Use the following inequality

$$\mathbf{P}\left(\max_{1 \le i \le n} |X_i| > \lambda\right) = \mathbf{P}(\cup_{i=1}^n \{|X_i| > \lambda\}) \le \sum_{i=1}^n \mathbf{P}(|X_i| > \lambda) = n\mathbf{P}(|X_i| > \lambda)$$

and a Markov type inequality.

11. **Homework 3.A (13th Apr 2022)** Let $X_1, X_2, \dots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \to \infty} x^\alpha (1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim b x^{-\alpha}$ as $x \to \infty$), then show that the distribution of $(bn)^{-1/\alpha} M_n$ converges weakly to the Fréchet distribution:

$$\mathbf{P}\left((bn)^{-1/\alpha} M_n < x\right) \to \mathbb{1}(x > 0) \exp\left(-x^{-\alpha}\right).$$

12. Let $X_1, X_2, \dots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \to \infty} e^{\lambda x}(1 - F(x)) = b$ for some fixed constants $\lambda, b \in (0, \infty)$ (that is, $1 - F(x) \sim b e^{-\lambda x}$ as $x \to \infty$), then show that the distribution of $\lambda M_n - \log(bn)$ converges weakly to the Gumbel distribution:

$$\mathbf{P}\left(\lambda M_n - \log(bn) < x\right) \to \exp\left(-e^{-x}\right).$$

13. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1\le i\le n} X_i$. If $F(x_0) = 1$ and $F(x) < 1$ for all $x < x_0$ and $\lim_{x\to x_0}(x_0 - x)^{-\alpha}(1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0,\infty)$ (that is, $1 - F(x) \sim b(x_0 - x)^\alpha$ as $x \to x_0$), then show that the distribution of $(bn)^{1/\alpha}(M_n - x_0)$ converges weakly to the Weibull distribution:

$$\mathbf{P}\left((bn)^{1/\alpha}(M_n - x_0) < x\right) \to \mathbb{1}(x < 0)\exp\left(-(-x)^\alpha\right) + \mathbb{1}(x \ge 0).$$

14. **Homework 3.B (13th Apr 2022)** Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Suppose that the common distribution has a finite right endpoint, that is,

$$x_F = \sup\{x \in \mathbb{R} : F(x) < 1\} < \infty$$

and that the right endpoint has a positive mass $\mathbf{P}(X_i = x_F) = \overline{F}(x_F) > 0$. Let $M_n := \max_{1\le i\le n} X_i$. Prove that for any sequence of reals (u_n), if the sequence of probabilities $\mathbf{P}(M_n < u_n)$ converges, then the limit is either 0 or 1.

Hint: Use the proposition about Poisson approximation to see that it is enough to show that $n\overline{F}(u_n)$ goes either to 0 or to ∞.

15. **Homework 3.C (13th Apr 2022)** Let X be a random variable with distribution function F where the tail is $1 - F(x) = x^{-\alpha}$ for $x \ge 1$ with some $\alpha > 0$. Then we know that $F \in \mathrm{MDA}(\Phi_\alpha)$. Which MDA does the distribution of X^p and that of $\ln(X)$ belong to if $p > 0$? What are the normalization constants?

16. Suppose that $X > 0$ is a random variable and $\alpha > 0$ is a number. Show that the following are equivalent:

    (a) X has the Fréchet distribution function

    $$\Phi_\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$$

    (b) $\ln X^\alpha$ has the Gumbel distribution function

    $$\Lambda(x) = \exp(-e^{-x})$$

    (c) $-X^{-1}$ has the Weibull distribution function

    $$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

    Show the equivalence of (a) and (b).

17. Let $X_1, X_2, \ldots$ be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k. Denote by $m_n = \min(X_1, \ldots, X_n)$ the minimum record up to n and let $m_\infty = \lim_{n\to\infty} m_n$ which exists since m_n is non-increasing.

    (a) Show that if $\sum_{k=1}^\infty \lambda_k < \infty$, then with probability one, the minimum record is broken finitely many times and $m_\infty > 0$.

    (b) Show that if $\sum_{k=1}^\infty \lambda_k = \infty$, then with probability one, the minimum record is broken infinitely many times and $m_\infty = 0$.

Hint: One can use the representation of X_k as the first point of a Poisson point process on $\mathbb{R}_+$ with intensity λ_k and the fact that the union of independent Poisson point processes with intensities λ_k for $k = 1, 2, \ldots$ is a Poisson point process with intensity $\sum_{k=1}^{\infty} \lambda_k$.

18. Let $X_1, X_2, \ldots$ be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k. Denote by $M_n = \max(X_1, \ldots, X_n)$ the maximum record up to n and let $M_\infty = \lim_{n\to\infty} M_n$ which exists since M_n is non-decreasing.

    (a) Show that for any constant $K > 0$ if

    $$\sum_{n=1}^{\infty} e^{-\lambda_n K} = \infty,$$

    then with probability one there are infinitely many indices n such that $X_n > K$. Further, for any $K > 0$ if

    $$\sum_{n=1}^{\infty} e^{-\lambda_n K} < \infty,$$

    then with probability one there are finitely many indices n such that $X_n > K$.

    Hint: Use the Borel–Cantelli lemmas.

    (b) Conclude that if $\sum_{n=1}^{\infty} e^{-\lambda_n K} = \infty$ for any $K > 0$, then with probability one, $M_\infty = \infty$ and the maximum record is broken infinitely many times. If $\sum_{n=1}^{\infty} e^{-\lambda_n K} < \infty$ for any $K > 0$, then with probability one, $M_\infty < \infty$ and the maximum record is broken finitely many times.

    (c) If there are $K_1 > K_2 > 0$ such that $\sum_{n=1}^{\infty} e^{-\lambda_n K_1} < \infty$ but $\sum_{n=1}^{\infty} e^{-\lambda_n K_2} = \infty$, then there is a critical K_c such that for smaller values of K the sum is infinite and for larger values of K the sum is finite. Show that in this case with probability one

    $$\limsup_{n\to\infty} X_n = K_c.$$

    (d) Suppose that there is a critical K_c as described above. Show that if

    $$\sum_{n=1}^{\infty} e^{-\lambda_n K_c} = \infty,$$

    then with probability one the maximum record is broken finitely many times.

    Hint: The condition shows that there are infinitely many indices n such that $X_n > K_c$, but for any $\varepsilon > 0$, there are only finitely many with $X_n > K_c + \varepsilon$.

    (e) Suppose that there is a critical K_c as described above. Show that if

    $$\sum_{n=1}^{\infty} e^{-\lambda_n K_c} < \infty,$$

    then with positive probability the maximum record is broken infinitely many times. Compute this probability.

    Hint: Observe that the maximum record is broken infinitely many times if and only if $X_n < K_c$ for all indices n, since $\limsup_{n\to\infty} X_n = K_c$.