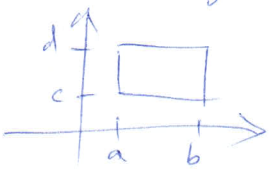
Multivariate extremes

Max-infinite divisibility

For a one-dim. distribution function $F(x)$ and $t>0$, $x \mapsto F^t(x)$ is also a distribution function. In higher dimensions, it might not be the case.

Example $d=2$



It is required that $F^t(b,d) - F^t(b,c) - F^t(a,d) + F^t(a,c)$ which is not automatic for any $t>0$.

Def

F is max-infinitely divisible (max-id) if $F^{1/n}$ is a distribution for any n.
Equivalently, if F^t is a distribution for every $t>0$.

Prop

Let F be a distribution on $\mathbb{R}^2$ with continuous density F''_{xy}. Then F is max-id iff $Q=-\log F$ satisfies $Q''_{xy} \leq 0$ on the set $\{F>0\}$.
Equivalently iff $F'_x F'_y \leq F''_{xy} F$ on $\mathbb{R}^2$.

Proof On $\{F>0\}$, we have $F^t = e^{-tQ}$ for $t>0$

$$(F^t)''_{xy} = \left(-te^{-tQ} Q'_x\right)'_y = -te^{-tQ}\left(Q''_{xy} - tQ'_x Q'_y\right) \geq 0 \iff F \text{ is max-id}$$

$$\iff Q''_{xy} - tQ'_x Q'_y \leq 0$$

Since $Q'_x = -\frac{F'_x}{F} \leq 0$ and $Q'_y \leq 0$, the condition holds iff $Q''_{xy} \leq 0$

$$Q''_{xy} = \left(-\frac{F'_x}{F}\right)'_y = -\frac{F''_{xy}F - F'_x F'_y}{F^2} \leq 0 \iff F'_x F'_y \leq F''_{xy} F$$

Example

Uniform distr. on $[0,1]^d$ is max-id

$$F^t(\{a_1,b_1\}\times\ldots\times\{a_d,b_d\}) = \prod_{j=1}^d (b_j^t - a_j^t)$$

$d=2$ by the Prop. $F(x,y)=xy$ $\quad Q = -\log x - \log y \quad Q''_{xy}=0$

Example (compound Poisson analogue)

$(U_n)_{n\ge1}$ iid. $\mathbb{R}^d$ valued rv's. s.t. $\exists\ \underline{L}\in\mathbb{R}^d$: $U_i \ge \underline{L}$ a.s.

Let $N(t)$ be a PPP on $(0,\infty)$ with unit intensity, independent of $(U_n)_{n\ge1}$.

Define $Y(t) = \max(U_i,\ i=1,\dots,N(t))$ and $Y(t)=\underline{L}$ if $N(t)=0$.

For any $\underline{x}\ge\underline{L}$, the distribution of $Y(t)$ is

$$F_t(\underline{x}) = \mathbb{P}(Y(t)\le\underline{x}) = \sum_{n=0}^\infty e^{-t}\frac{t^n}{n!}\underbrace{\mathbb{P}(\max(U_1,\dots,U_n)\le\underline{x})}_{P(U_1\le\underline{x})^n}$$

$$= \exp(-t(1-P(U_1\le\underline{x}))) = \exp(-t\,\mathbb{P}(\{U_1\le\underline{x}\}^c)). \text{ is max-id.}$$

Example

Let $\underline{L}\in[-\infty,\infty)^d$ and consider $E=[\underline{L},\infty]\setminus\{\underline{L}\}$

Suppose that μ is a Radon measure on E (i.e. locally finite and inner reg

$\mu(B)=\sup\{\mu(K): K\subseteq B \text{ comp}$

Let $\xi = \sum_k \delta_{(t_k,j_k)}$ be a PPP on $\mathbb{R}_+\times E$ with mean measure $dt\times d\mu$

Define $Y(t) = \max(j_k,\ t_k\le t)$ and $Y(t)=\underline{L}$ if there is no such k

Then for any $\underline{x}\ge\underline{L}$, the distribution of $Y(t)$ is

$$F_t(\underline{x}) = P(Y(t)\le\underline{x}) = \mathbb{P}(\xi \text{ has no points in } [0,t]\times[-\infty,\underline{x}]^{\overset{\text{w.r.t. } E}{c}})$$

$$= \exp(-t\,\mu([-\infty,\underline{x}]^c)) \quad \text{is max-id.}$$

Then μ is called the exponent measure.

For example, if $F(\underline{x}) = \prod_{i=1}^d \Lambda(x_i)$ where $\Lambda(x)=\exp(-e^{-x})$, then

$$\mu\Big(\bigcup_{i,j\in\{1,\dots,d\}}\{\underline{x}\in[-\infty,\infty]^d\setminus\{-\underline{\infty}\} : x_i>-\infty,\ x_j>-\infty\}\Big)=0$$

$$\mu((x,\infty]\times\{-\infty\}\times\ldots\times\{-\infty\}) = \ldots = \mu(\{-\infty\}\times\ldots\times\{-\infty\}\times(x,\infty]) = e^{-x}$$

which vanishes on the interior of $[\underline{L},\infty] = [-\infty,\infty]^d$

Then $\mu([-\infty,\underline{x}]^c) = \mu\Big(\bigcup_{i=1}^d\{\underline{y}: y_i>x_i\}\Big) = \sum_{i=1}^d \mu(\{(-\infty,\dots,-\infty,y_i,-\infty,\dots,-\infty): y_i>x_i\}) = \sum_{i=1}^d e^{-x_i}$

and $F(\underline{x}) = \exp(-\mu[-\infty,\underline{x}]^c) = \exp(-\sum_{i=1}^d e^{-x_i}) = \prod_{i=1}^d \Lambda(x_i)$.

Thm (characterization of max-id distributions)
The following are equivalent:
1. F is max-id
2. For some $\underline{L} \in [-\infty, \infty)^d$, there is an exponent measure μ on $E = [\underline{L}, \infty]$ such that μ is a Radon measure with extra technical assumptions so that

$$F(\underline{x}) = \exp(-\mu([-\infty, \underline{x}]^c)) \text{ if } \underline{x} \geq \underline{L} \text{ and } 0 \text{ otherwise.}$$

Multivariate extreme value distributions

$\underline{X}_n = (X_n^{(1)}, \dots, X_n^{(d)}) \in \mathbb{R}^d$ r.v's $n \geq 1$ i.i.d. with distribution $F(x)$.

If there are $a_n^{(i)} > 0$ and $b_n^{(i)} \in \mathbb{R}$ $i = 1, \dots, d$ and $n = 1, 2, \dots$ s.t.

$$\mathbb{P}\left(\frac{M_n^{(i)} - b_n^{(i)}}{a_n^{(i)}} < x^{(i)} \quad i = 1, \dots, d\right) = F^n(a_n^{(1)} x^{(1)} + b_n^{(1)}, \dots, a_n^{(d)} x^{(d)} + b_n^{(d)}) \to G(\underline{x})$$

so that each marginal of G is non-degenerate, then G is called a multivariate extreme value distribution. F is in the domain of attraction of
For the ith component, $F_i^n(a_n^{(i)} x + b_n^{(i)}) \to G_i(x)$ $1 \leq i \leq d$ where G_i is one of $\Phi_\alpha, \Psi_\alpha, \Lambda$.

Def
The distribution $G(\underline{x})$ is max-stable if there are $\alpha^{(i)}(t) > 0$, $\beta^{(i)}(t) \in$ for $t > 0$ and $i = 1, \dots, d$ s.t.

$$G^t(\underline{x}) = G(\alpha^{(1)}(t) x^{(1)} + \beta^{(1)}, \dots, \alpha^{(d)}(t) x^{(d)} + \beta^{(d)}) \quad \text{for any } t > 0.$$

Thm
G is a multivariate extreme value distribution $\Leftrightarrow$ G is max-stable with non-degenerate marginals.

Proof is the ~~similar to~~ $d = 1$, $\boxed{\Leftarrow}$ is obvious, $\boxed{\Rightarrow}$ follows by convergence of

Prop
Suppose G is a multivariate distribution with continuous marginals. Define
$\Psi_i(x) = \left(\frac{1}{-\log G_i}\right)^{-1}(x)$ for $x > 0$ and $i = 1, \dots, d$ and $G_*(\underline{x}) = G(\Psi_1(x^{(1)}), \dots \Psi_d(x^{(d)}))$
Then G_* has marginals $G_{*i}(x) = \Phi_1(x)$ and G is a multivariate extreme value distribution iff G_* is.
Further, let $U_i = 1/(1-F_i)$ $i = 1, \dots, d$ and let F_* be the distribution of $(U_1(X_1^{(1)}), \dots, U_d$
i.e. $F_*(\underline{x}) = F(U_1^{-1}(x^{(1)}), \dots, U_d^{-1}(x^{(d)}))$. Then

$$\mathbb{P}\left(\frac{M_n^{(i)} - b_n^{(i)}}{a_n^{(i)}} < x^{(i)} \quad i = 1, \dots, d\right) = F^n(a_n^{(1)} x^{(1)} + b_n^{(1)}, \dots, a_n^{(d)} x^{(d)} + b_n^{(d)}) \to G(\underline{x}) \text{ if and only if}$$

$$\mathbb{P}\left(\max_{j=1}^{n} (U_i(X_j^{(i)})/n) < x^{(i)} \quad i = 1, \dots, d\right) = F_*^n(n\underline{x}) \to G_*(\underline{x}).$$

Characterization of max-stable distributions

Suppose that G_* has Φ_1 marginals, hence $a_n^{(i)} = n$ and $b_n^{(i)} = 0$ can be chosen.
If G_* is max-stable, then in the definition $\alpha^{(i)}(t) = \frac{1}{t}$ and $\beta^{(i)}(t) = 0$ can be chosen.
Hence $G_*^t(t\underline{x}) = G_*(\underline{x})$. Note that G_* is also max-id.
Since the marginals are concentrated on $[0,\infty)$, the exponent measure μ_* concentrates on $E = [0,\infty]^d \setminus \{\underline{0}\}$. Since $G_*(\underline{x}) = \exp(-\mu([0,\underline{x}]^c))$, we have
$$\mu_*([0,\underline{x}]^c) = t\,\mu_*([0,t\underline{x}]^c) = t\,\mu_*(t[0,\underline{x}]^c)$$
This implies that for any rectangle $B \subseteq E$, one has
$$\mu_*(B) = t\,\mu_*(tB)$$
which extends to all Borel subsets of E.

Let $S = \{y \in E : \|y\| = 1\}$. The choice of the norm is arbitrary, since all norms in $\mathbb{R}^d$ are equivalent.
For Borel subsets $A \subseteq S$, define the measure
$$\nu(A) = \mu_*(\{\underline{x} : \|\underline{x}\| > 1, \|\underline{x}\|^{-1}\underline{x} \in A\}).$$
Since μ_* is finite on compact sets, $\{\underline{x} : \|\underline{x}\| > 1, \|\underline{x}\|^{-1}\underline{x} \in A\} \subseteq \bigcup_{k=0}^{\infty} A_k$ with $A_k = \{\underline{x} : 2^k \le \|\underline{x}\| \le 2^{k+1}, \|\underline{x}\|^{-1}\underline{x} \in A\}$ has finite measure and $\mu_*(A_k) = 2^{-k}\mu_*(A_0)$.
Similarly, $\mu_*(\{y \in E : \|y\| > r, \|y\|^{-1}y \in A\}) = r^{-1}\mu_*(\{y \in E : \|y\| > 1, \|y\|^{-1}y \in A\}) = r^{-1}\nu$
Hence in the polar coordinates $(\|y\|, \|y\|^{-1}y)$, μ_* is a product measure
$T : E \to (0,\infty] \times S$ $\quad Ty = (\|y\|, \|y\|^{-1}y)$ allows us to write
$$\mu_* \circ T^{-1}(dr, d\underline{a}) = r^{-2}\,dr\,\nu(d\underline{a})$$
Hence
$$\begin{aligned}\mu_*([0,\underline{x}]^c) &= \mu_* \circ T^{-1} \circ T([0,x]^c) \\ &= \mu_* \circ T^{-1}(\{(r,\underline{a}) \in ((0,\infty] \times S) : (r\underline{a})^{(i)} > x^{(i)} \text{ for some } i\}) \\ &= \mu_* \circ T^{-1}(\{(r,\underline{a}) : r > \min(\tfrac{x^{(i)}}{a^{(i)}}, i=1,\dots,d)\}) \\ &= \int_S \nu(d\underline{a}) \int_{\{r > \min(\frac{x^{(i)}}{a^{(i)}}, i=1,\dots,d)\}} r^{2}\,dr \\ &= \int_S \nu(d\underline{a}) \left(\min\left(\tfrac{x^{(i)}}{a^{(i)}}, i=1,\dots,d\right)\right)^{-1} = \int_S \max\left(\tfrac{a^{(i)}}{x^{(i)}}, i=1,\dots,d\right)\nu(d\underline{a})\end{aligned}$$

Thm

The following are equivalent:

1. $G_*(\underline{x})$ is a multivariate extreme value distribution with Φ_1 marginals
2. There is a finite measure ν on $S = \{ \underline{y} \in E : \|\underline{y}\| = 1 \}$
   with $\int_S a^{(i)} \, \nu(d\underline{a}) = 1$ for $i = 1, \ldots, d$
   such that for any $\underline{x} \in \mathbb{R}^d$,
   $$G_*(\underline{x}) = \exp\left(- \int_S \max\left(\frac{a^{(i)}}{x^{(i)}}, i = 1, \ldots, d \right) \nu(d\underline{a}) \right).$$

Remark

The condition $\int_S a^{(i)} \, \nu(d\underline{a}) = 1$ guaranties the Φ_1 marginals.