Extreme value theory midterm exam, 25 April 2018

1. (a) What does it mean that a function is slowly varying at infinity?

   (b) Consider the function $f(x) = e^{\sqrt{\log x}}$. Show that it grows at infinity slower than any polynomial, that is, $\lim_{x\to\infty} f(x)/x^\varepsilon = 0$ for any fixed $\varepsilon > 0$.

   (c) Show that $f(x)$ grows at infinity faster than any power of the logarithm, that is, $\lim_{x\to\infty} (\log x)^k / f(x) = 0$ for any fixed $k > 0$.

   (d) Is $f(x)$ slowly varying at infinity?

2. Let $\tau = \inf\{t > 0 : B_t = 1\}$ be the hitting time of level 1 by the standard Brownian motion B_t which starts from $B_0 = 0$. Show that τ has the Lévy distribution, that is, its density is $(2\pi x^3)^{-1/2} \exp(-1/(2x))$ for $x > 0$.

   Hint: It can be used without proof that as a consequence of the reflection principle

   $$\mathbf{P}\left(\max\{B_s : s \in [0,t]\} > x\right) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

   where $\Phi(x) = \int_{-\infty}^x (2\pi)^{-1/2} \exp(-y^2/2)\,\mathrm{d}y$ is the normal distribution function. The distribution function of τ can be directly expressed by the probability in the above equality for $x = 1$.

3. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1\le i\le n} X_i$. If $F(x_0) = 1$ and $F(x) < 1$ for all $x < x_0$ and $\lim_{x\to x_0} (x_0 - x)^{-\alpha}(1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0,\infty)$ (that is, $1 - F(x) \sim b(x_0 - x)^\alpha$ as $x \to x_0$), then show that the distribution of $(bn)^{1/\alpha}(M_n - x_0)$ converges weakly to the Weibull distribution:

   $$\mathbf{P}\left((bn)^{1/\alpha}(M_n - x_0) < x\right) \to \mathbb{1}(x < 0)\exp\left(-(-x)^\alpha\right) + \mathbb{1}(x \ge 0).$$

4. Let $X_1, X_2, \ldots$ be an iid sequence of Poisson random variables with parameter $\lambda > 0$, that is,

   $$\mathbf{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

   for $k = 0, 1, 2, \ldots$ and let $M_n := \max_{1\le i\le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

   Hint: Show that the necessary condition

   $$\lim_{x\uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

   fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converge to 0 for the integers $k \to \infty$.

5. Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed uniform random variables on $[0,1]$. Let $X_k = 1/U_k^3$ for $k = 1, 2, \ldots$ and form $M_n = \max(X_1, \ldots, X_n)$. Which non-trivial limit distribution does the renormalized sequence of M_n converge to? Under what normalization?

6. (a) What is a Poisson point process with given mean measure (or intensity) on $(0, \infty)$? How can the homogeneous Poisson point process (i.e. Lebesgue mean measure or constant intensity) be interpreted with iid. exponential random variables?

   (b) What is the limit of the sequence of measures associated to rescaled record times observed in an iid. sequence with continuous distribution?

7. Let $X_1, X_2, \ldots$ be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k. Denote by $m_n = \min(X_1, \ldots, X_n)$ the minimum record up to n and let $m_\infty = \lim_{n\to\infty} m_n$ which exists since m_n is non-increasing.

   (a) Show that if $\sum_{k=1}^{\infty} \lambda_k < \infty$, then with probability one, the minimum record is broken finitely many times and $m_\infty > 0$.

   (b) Show that if $\sum_{k=1}^{\infty} \lambda_k = \infty$, then with probability one, the minimum record is broken infinitely many times and $m_\infty = 0$.

   Hint: One can use the representation of X_k as the first point of a Poisson point process on $\mathbb{R}_+$ with intensity λ_k and the fact that the union of independent Poisson point processes with intensities λ_k for $k = 1, 2, \ldots$ is a Poisson point process with intensity $\sum_{k=1}^{\infty} \lambda_k$.