Domains of attraction for sums, stable laws

Convergence in distribution

Def

The sequence X_n of random variables with values in $\mathbb{R}^d$ converges in distribution to X if for all $f: \mathbb{R}^d \to \mathbb{R}$ bounded and continuous
$$\mathbb{E}(f(X_n)) \longrightarrow \mathbb{E}(f(X)) \quad \text{as } n \to \infty.$$
Notation: $X_n \xrightarrow{d} X$.

Proposition

Let X_n and X be real random variables and let $F_n(x) = \mathbb{P}(X_n < x)$ and $F(x) = \mathbb{P}(X < x)$ be their distribution function.

Then $X_n \xrightarrow{d} X$ if and only if $F_n(x) \to F(x)$ for all x which are points of continuity of F.

Central limit theorem (CLT)

Let $(X_n)_{n=1}^{\infty}$ be independent identically distributed (iid) random variables. Suppose that $\mathbb{E}(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$.

For $S_n = X_1 + \ldots + X_n$,
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{X}$$
where $\mathcal{X} \sim N(0,1)$.

Characteristic function

For a random variable X, let its characteristic function be
$$\varphi(t) = \mathbb{E}(e^{itX}) \quad \text{for any } t \in \mathbb{R}.$$

Properties:

- the expectation exists for all $t \in \mathbb{R}$ and $|\varphi(t)| \le 1$
- the same as the Fourier transform of the distribution (measure)/density
- as a consequence, the characteristic function determines the distribution
- Continuity thm

  If $X_n \xrightarrow{d} X$ and $\varphi_n(t) = \mathbb{E}(e^{itX_n})$ and $\varphi(t) = \mathbb{E}(e^{itX})$ then $\varphi_n(t) \to \varphi(t)$ for all $t \in \mathbb{R}$. If $\varphi_n(t) \to \varphi(t)$ for all $t \in \mathbb{R}$ where $\varphi_n(t) = \mathbb{E}(e^{itX_n})$ and $\varphi(t)$ is continuous at 0, then there is an X s.t. $X_n \xrightarrow{d} X$ and $\varphi(t) = \mathbb{E}(e^{itX})$.

- if $X \sim N(0,1)$, then $\varphi(t) = \mathbb{E}(e^{itX}) = e^{-t^2/2}$.
- $\varphi(t) = \mathbb{E}(e^{itX}) = 1 + it\,\mathbb{E}X - \frac{t^2\,\mathbb{E}(X^2)}{2} + o(t^2)$ if $\mathbb{E}X, \mathbb{E}(X^2) < \infty$.

Proof of ~~the CLT~~

Assume $\mathbb{E}X_n = 0$.

We compute

$$\mathbb{E}\left(e^{it\frac{S_n}{\sigma\sqrt{n}}}\right) = \mathbb{E}\left(e^{i\frac{t}{\sigma\sqrt{n}}(X_1+\dots+X_n)}\right) \underset{\text{indep.}}{=} \prod_{k=1}^{n} \mathbb{E}\left(e^{i\frac{t}{\sigma\sqrt{n}}X_1}\right)$$

$$= \left(\mathbb{E}\left(e^{i\frac{t}{\sigma\sqrt{n}}X_1}\right)\right)^n = \left(1 - \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2 \mathbb{E}(X_1^2)}{2} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-\frac{t^2}{2}}. \quad \square$$

Domain of attraction

Def

The distribution function F is in the domain of attraction of the distribution ϕ if $\exists\, a_n > 0$ and $b_n \in \mathbb{R}$ s.t. if $X_1, X_2, \dots$ are iid with distribution F and $S_n = X_1 + \dots + X_n$, then $\frac{S_n - b_n}{a_n} \xrightarrow{d} \psi$ where ψ has distribution ϕ. Notation: $F \in DA(\phi)$.

Thm

The distribution F is in the domain of attraction of the normal distr, iff

1) F has finite variance (in which case the CLT applies) or
2) F satisfies the condition
$$\lim_{x\to\infty} \frac{x^2\,\mathbb{P}(|X|>x)}{\int_{|y|\le x} y^2\, dF(y)} = 0 \qquad (*)$$

where X has distribution F.

Exercise

Show that if F has finite variance, then the condition $(*)$ holds, i.e. in the thm above, condition 1) implies 2).

Stable laws

Goal: find other non-degenerate random variable Y s.t. there is an iid. sequence $X_1, X_2, \dots$ with $S_n = X_1 + \dots + X_n$ and $a_n > 0$ and $b_n \in \mathbb{R}$ for which $\frac{S_n - b_n}{a_n} \xrightarrow{d} Y$ holds.

Example (symmetric)

Let $X_1, X_2, \dots$ be an iid. sequnce with distribution

$\mathbb{P}(|X_1| > x) = x^{-\alpha}$ for $x \geq 1$ and $\mathbb{P}(X_1 > x) = \mathbb{P}(X_1 < -x)$.

where $\alpha \in (0,2)$. Density: $f(x) = \frac{\alpha}{2} |x|^{-\alpha - 1}$, $|x| > 1$

Characteristic function: $\varphi(t) = \mathbb{E}(e^{itX_1})$

For $t > 0$, $1 - \varphi(t) = \int_1^{\infty} (1 - e^{itx}) \frac{\alpha}{2x^{\alpha+1}} dx + \int_{-\infty}^{-1} (1 - e^{itx}) \frac{\alpha}{2|x|^{\alpha+1}} dx$

$$= \int_1^{\infty} \left(1 - \frac{e^{itx} + e^{-itx}}{2}\right) \frac{\alpha}{x^{\alpha+1}} dx = \alpha \int_1^{\infty} \frac{1 - \cos(tx)}{x^{\alpha+1}} dx$$

$$\underset{u = tx}{=} \alpha t^{\alpha} \int_t^{\infty} \frac{1 - \cos u}{u^{\alpha+1}} du$$

- as $u \to \infty$, the integrand is $\asymp \frac{1}{u^{\alpha+1}}$ integrable for $\alpha \in (0,2)$
- as $u \to 0$. $\frac{1 - \cos u}{u^{\alpha+1}} \asymp \frac{u^2/2}{u^{\alpha+1}} = \frac{1}{2} u^{1-\alpha}$ integrable for $\alpha \in (0,2)$

Let $C = \alpha \int_0^{\infty} \frac{1 - \cos u}{u^{\alpha+1}} du$. Then $1 - \varphi(t) = C|t|^{\alpha} + o(|t|^{\alpha})$ as $t \to 0$ since $\varphi(-t) = \varphi(t)$ by the symmetry of the distribution.

Let $S_n = X_1 + \dots + X_n$. We show that $S_n / n^{1/\alpha}$ converges in distr:

$$\mathbb{E}\left(e^{it \frac{S_n}{n^{1/\alpha}}}\right) \underset{\text{iid.}}{=} \left(\mathbb{E}\left(e^{i \frac{t}{n^{1/\alpha}} X_1}\right)\right)^n = \varphi\left(\frac{t}{n^{1/\alpha}}\right)^n = \left(1 - \frac{C|t|^{\alpha}}{n} + o\left(\frac{|t|^{\alpha}}{n}\right)\right)^n$$

$\longrightarrow e^{-C|t|^{\alpha}}$ as $n \to \infty$ which is continuous at $t = 0$.

By the continuity thm, $\frac{S_n}{n^{1/\alpha}} \xrightarrow{d} Y$ and $\mathbb{E}(e^{itY}) = e^{-C|t|^{\alpha}}$. $\square$

Remarks

- In the example above, we assumed $\alpha \in (0,2)$. For $\alpha > 2$, the variance is finite $\Rightarrow$ CLT holds.
- For $\alpha = 2$, the variance is infinite, but condition $(*)$ holds. Actually, $\frac{S_n}{\sqrt{n \log n}} \xrightarrow{d} N(0, \sigma^2)$ can be seen by characteristic function.
- For $\alpha \in (0,1)$, $n^{1/\alpha} >> n$, hence the main contribution of S_n comes from a few terms of size $n^{1/\alpha}$.

Def
The distribution of the random variables X and Y is of the same type if $\exists a > 0$ and $b \in \mathbb{R}$ s.t. $aX + b \overset{d}{=} Y$.

Def
Y has a stable law if for every positive integer k, $\exists a_k > 0$ and $b_k \in \mathbb{R}$ s.t. if $Y_1, \dots, Y_k$ are iid. with the distr. of Y, then $\frac{Y_1 + \dots + Y_k - b_k}{a_k} \overset{d}{=} Y$, i.e. $Y_1 + \dots + Y_k$ and Y have distributions of the same type.

Thm
The symmetric stable laws are those which appeared in the example, i.e. the ones with characteristic function $e^{-c|t|^\alpha}$ for index $\alpha \in (0,2]$.

Def
The function $L: \mathbb{R} \to \mathbb{R}$ is slowly varying if $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$ for all $t > 0$.

Thm (non-symmetric)
Let $X_1, X_2, \dots$ iid with distribution $\mathbb{P}(|X_1| > x) = x^{-\alpha} L(x)$ for some index $\alpha \in (0,2)$ and slowly varying L and skewness $\kappa = \lim_{x \to \infty} \frac{P(X_1 > x) - P(X_1 < -x)}{P(|X_1| > x)} \in [-1,1]$.
Let $S_n = X_1 + \dots + X_n$, $a_n = \inf\{x : P(|X_1| > x) \le \frac{1}{n}\}$ and $b_n = n \mathbb{E}(X_1 \mathbb{1}(|X_1| \le a_n))$.
Then $\frac{S_n - b_n}{a_n} \xrightarrow{d} Y$ as $n \to \infty$ where Y is non-degenerate.
The distribution of Y is stable with index α and skewness κ.
There are no other stable laws.

Example

Let $\alpha \in (0,2)$ and $K \in [-1,1]$.

$$\left.\begin{aligned} P(X_1 > x) &= \frac{1+K}{2} x^{-\alpha} \\ P(X_1 < -x) &= \frac{1-K}{2} x^{-\alpha} \end{aligned}\right\} \text{ for } x \geq 1 \qquad \text{density } f(x) = \frac{1 \pm K}{2} \alpha |x|^{-\alpha-1} \text{ for } |x| \geq 1$$

By the thm above, $a_n = n^{1/\alpha}$ and $b_n = n \int_1^{n^{1/\alpha}} x \left(f(x) - f(-x)\right) dx$

$$= n \int_1^{n^{1/\alpha}} K\alpha x^{-\alpha} dx \asymp \begin{cases} n & \alpha > 1 \\ n \log n & \alpha = 1 \\ n^{1/\alpha} & \alpha < 1 \end{cases}$$

Special stable laws (no explicit formulas otherwise)

$\alpha = 2$ normal distr. (no K dependence) $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$\alpha = 1$, $K = 0$ Cauchy distr. $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

$\alpha = 1/2$, $K = 1$ Lévy distribution $f(x) = \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{1}{2x}}$ for $x > 0$

(appears as hitting time distribution for BM)

Thm

Y has stable law iff Y appears as the limit of $\frac{X_1 + \dots + X_n - b_n}{a_n}$ for some iid. $X_1, X_2, \dots$

Thm (convergence of types)

Suppose that $\frac{X_n - b_n}{a_n} \overset{d}{\Rightarrow} U$ and $\frac{X_n - \beta_n}{\alpha_n} \overset{d}{\Rightarrow} V$

for non-degenerate rvs X_n, U, V; $a_n, \alpha_n > 0$; $b_n, \beta_n \in \mathbb{R}$.

Then $\frac{\alpha_n}{a_n} \to A > 0$ and $\frac{\beta_n - b_n}{a_n} \to B \in \mathbb{R}$ and $V \overset{d}{=} \frac{U - B}{A}$.

Proof of thm on stable laws

$\boxed{\Rightarrow}$ Let $X_1, X_2, \dots$ be iid. with the stable law of Y.

$\boxed{\Leftarrow}$ Fix an integer k and let $S_{nk} = S_n^{(1)} + \dots + S_n^{(k)}$

where $S_n^{(j)} = X_{(j-1)n+1} + \dots + X_{jn}$ for which $\frac{S_n^{(j)} - b_n}{a_n} \overset{d}{\Rightarrow} Y$

Hence $\frac{S_{nk} - k b_n}{a_n} = \frac{S_n^{(1)} - b_n}{a_n} + \dots + \frac{S_n^{(k)} - b_n}{a_n} \overset{d}{\Rightarrow} Y_1 + \dots + Y_k$ where $Y_1, \dots, Y_k$ are iid with distr. of Y

On the other hand, $\frac{S_{nk} - b_{nk}}{a_{nk}} \overset{d}{\Rightarrow} Y$.

By convergence of types: $\frac{Y_1 + \dots + Y_k - B_k}{A_k} \overset{d}{=} Y$ where $A_k = \lim_{n\to\infty} \frac{a_{nk}}{a_n}$, $B_k = \lim_{n\to\infty} \frac{b_{nk} - k b_n}{a_n}$.

$\square$

If $H: \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then let $H^{-1}(y) = \inf\{s : H(s) > y\}$ be its right continuous inverse.

<u>Prop</u> (technical)

If H_n are non-decreasing and $H_n \to H$ for all points of continuity of H, then $H_n^{-1} \to H^{-1}$ for all points of continuity of H^{-1}.

<u>Proof of convergence of types thm</u>

With the notation $F_n(x) = \mathbb{P}(X_n < x)$, $U(x) = \mathbb{P}(U < x)$, $V(x) = \mathbb{P}(V < x)$, we have $F_n(a_n x + b_n) \to U(x)$ for all points of continuity of U.

By the prop. above and by taking inverses:

$$\frac{F_n^{-1}(y) - b_n}{a_n} \to U^{-1}(y) \quad \text{and similarly} \quad \frac{F_n^{-1}(y) - \beta_n}{\alpha_n} \to V^{-1}(y)$$

for all points of continuity of the RHS.

Let $y_1 < y_2$ with $U^{-1}(y_1) < U^{-1}(y_2)$ and $V^{-1}(y_1) < V^{-1}(y_2)$.

Then $$\frac{F_n^{-1}(y_i) - b_n}{a_n} \to U^{-1}(y_i) \quad \text{and} \quad \frac{F_n^{-1}(y_i) - \beta_n}{\alpha_n} \to V^{-1}(y_i) \quad i = 1,2$$

By subtraction,

$$\frac{F_n^{-1}(y_2) - F_n^{-1}(y_1)}{a_n} \to U^{-1}(y_2) - U^{-1}(y_1) > 0 \quad \text{and} \quad \frac{F_n^{-1}(y_2) - F_n^{-1}(y_1)}{\alpha_n} \to V^{-1}(y_2) - V^{-1}(y_1) > 0$$

By division, $$\frac{\alpha_n}{a_n} \to \frac{U^{-1}(y_2) - U^{-1}(y_1)}{V^{-1}(y_2) - V^{-1}(y_1)} =: A > 0$$

With this, $$\frac{F_n^{-1}(y_1) - b_n}{a_n} \to U^{-1}(y_1) \quad \text{and} \quad \frac{F_n^{-1}(y_1) - \beta_n}{a_n} \to A \cdot V^{-1}(y_1)$$

By subtraction, $$\frac{\beta_n - b_n}{a_n} \to U^{-1}(y_1) - AV^{-1}(y_1) =: B$$

Since $$\frac{X_n - b_n}{a_n} = \frac{\alpha_n}{a_n} \frac{X_n - \beta_n}{\alpha_n} + \frac{\beta_n - b_n}{a_n} \overset{d}{\Rightarrow} AV + B \quad \text{and} \quad \frac{X_n - b_n}{a_n} \Rightarrow U,$$

$AV + B \overset{d}{=} U$ as required. $\square$