Extreme value theory midterm exam, 19 April 2016

1. (10 points) Let X have a symmetric stable law with index α. Show that $\mathbf{E}|X|^p < \infty$ for $p < \alpha$.

   Hint: If $\varphi(t) = \mathbf{E}(e^{itX})$ denotes the characteristic function of X, then the following estimate can be used without proof for $u > 0$:

   $$\mathbf{P}\left(|X| > \frac{2}{u}\right) \leq \frac{1}{u} \int_{-u}^{u} (1 - \varphi(t))\,\mathrm{d}t$$

   and the identity

   $$\mathbf{E}(|X|^p) = p \int_0^\infty t^{p-1} \mathbf{P}(|X| > t)\,\mathrm{d}t.$$

2. (10 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Suppose that the common distribution has a finite right endpoint, that is,

   $$x_F = \sup\{x \in \mathbb{R} : F(x) < 1\} < \infty$$

   and that the right endpoint has a positive mass $\mathbf{P}(X_i = x_F) = \overline{F}(x_F) > 0$. Let $M_n := \max_{1 \leq i \leq n} X_i$. Prove that for any sequence of reals (u_n), if the sequence of probabilities $\mathbf{P}(M_n < u_n)$ converges, then the limit is either 0 or 1.

   Hint: Use the proposition about Poisson approximation.

3. (10 points) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \leq i \leq n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \to \infty} e^{\lambda x}(1 - F(x)) = b$ for some fixed constants $\lambda, b \in (0, \infty)$ (that is, $1 - F(x) \sim b e^{-\lambda x}$ as $x \to \infty$), then show that the distribution of $\lambda M_n - \log(bn)$ converges weakly to the Gumbel distribution:

   $$\mathbf{P}\left(\lambda M_n - \log(bn) < x\right) \to \exp\left(-e^{-x}\right).$$

4. (10 points)

   (a) Let X and Y be independent random variables with uniform distribution on $[0, 1]$. What is the distribution of $Z = X + Y$?

   (b) Suppose that $Z_1, Z_2, \ldots$ is an independent and identically distributed sequence with the distribution of Z from the previous part. Denote $M_n = \max(Z_1, \ldots, Z_n)$. Which extreme value distribution is the limit of the rescaled sequence of M_n? Under what normalization?

   Hint: How does the tail probability behave at the right endpoint of the distribution?

5. (10 points) Suppose that $X > 0$ is a random variable and $\alpha > 0$ is a number. Show that X has the Weibull distribution function

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

if and only if $-\alpha \ln(-X)$ has the Gumbel distribution function $\Lambda(x) = \exp(-e^{-x})$.

6. (15 points) Let $X_1, X_2, \dots$ be an independent but not identically distributed sequence of random variables, let X_k be exponential with parameter λ_k. Denote by $M_n = \max(X_1, \dots, X_n)$ the maximum record up to n.

   (a) Show that for any constant $K > 0$ if

   $$\sum_{n=1}^{\infty} e^{-\lambda_n K} = \infty,$$

   then with probability one there are infinitely many indices n such that $X_n > K$. Further, for any $K > 0$ if

   $$\sum_{n=1}^{\infty} e^{-\lambda_n K} < \infty,$$

   then with probability one there are finitely many indices n such that $X_n > K$.

   Hint: Use the Borel–Cantelli lemmas.

   (b) Suppose that there is a critical K_c such that for $K > K_c$, $\sum_{n=1}^{\infty} e^{-\lambda_n K_1} < \infty$ but for $K < K_c$, $\sum_{n=1}^{\infty} e^{-\lambda_n K_2} = \infty$. Show that in this case with probability one

   $$\limsup_{n\to\infty} X_n = K_c.$$

   (c) Let $\lambda_n = \ln n$. What is the value of K_c? Show that with probability one the maximum record is broken finitely many times.

   Hint: Check the finiteness of the sum $\sum_{n=1}^{\infty} e^{-\lambda_n K_c}$ and conclude that there are infinitely many indices n such that $X_n > K_c$, but for any $\varepsilon > 0$, there are only finitely many with $X_n > K_c + \varepsilon$.