Extreme value theory midterm exam, 26 March 2014

1. (a) What does it mean that the function $L(t)$ is slowly varying?

   (b) Show that $L_1(t) = \log^2 t$ is a slowly varying function but $L_2(t) = e^{\sqrt{x}}$ is not.

2. (a) What is a max-stable law?

   (b) How does the Fisher – Tippett theorem characterize the max-stable laws?

3. Suppose that X_n is a sequence of random variables and there are two sequences of positive reals a_n and α_n such that

$$\frac{X_n}{a_n} \xrightarrow{\mathrm{d}} U \quad \text{and} \quad \frac{X_n}{\alpha_n} \xrightarrow{\mathrm{d}} V$$

   where U and V are non-degenerate random variables. What is the relation of U and V?

   Hint: A proposition which we have leart can be used without proof.

4. Let X be a random variable with Fréchet distribution, i.e. with distribution function

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$$

   where $\alpha > 0$. Prove that $\ln(X^\alpha)$ has Gumbel distribution, i.e. its distribution function is

$$\Lambda(x) = \exp(-e^{-x}).$$

5. For $f : \mathbb{R} \to \mathbb{R}$ functions, consider Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

   Prove that the only solutions for the equation are the linear functions $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$, if we assume that f is bounded on the interval $[1, 2]$.

6. Let $X_1, X_2, \dots$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which have the same distribution function $F(x)$, but we do not assume anything about their dependence structure. Let $M_n := \max_{1 \le i \le n} X_i$. Suppose that there is an $\alpha > 0$ such that $\int_{-\infty}^{\infty} |x|^\alpha \mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(|X_i|^\alpha) < \infty$. Prove that for any $\varepsilon > 0$ and for fixed $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(n^{-(1/\alpha+\varepsilon)}|M_n| > \delta\right) = 0.$$

   Hint: Use the following inequality

$$\mathbf{P}\left(\max_{1\le i\le n} |X_i| > \lambda\right) = \mathbf{P}(\cup_{i=1}^n \{|X_i| > \lambda\}) \le \sum_{i=1}^n \mathbf{P}(|X_i| > \lambda) = n\mathbf{P}(|X_i| > \lambda)$$

   and Markov's inequality for the αth moment, that is,

$$\mathbf{P}(|X| > \lambda) \le \frac{\mathbf{E}(|X|^\alpha)}{\lambda^\alpha}.$$

7. Let $X_1, X_2, \dots$ be an iid sequence of Poisson random variables with parameter $\lambda > 0$, that is,

$$\mathbf{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \dots$ and let $M_n := \max_{1\le i\le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x\uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converge to 0 for the integers $k \to \infty$.

8. Let $X_1, X_2, \dots$ be i.i.d. uniform random variables on $[0, 1]$. Prove a limit theorem for their maximum $M_n := \max_{1\le i\le n} X_i$.

Hint: The following question helps to guess the correct rescaling of M_n. How does the tail probability behave at the right endpoint of the distribution?

9. Suppose that (u_n) is a sequence of real numbers such that $n(1-F(u_n)) \to \tau \in (0, \infty)$. Let $X_1, X_2, \dots$ be an i.i.d. sequence with distribution function F and let $M_n^{(2)}$ be the second largest value in the set $\{X_1, \dots, X_n\}$. What is $\lim_{n\to\infty} \mathbf{P}(M_n^{(2)} < u_n)$?

Hint: Let $S_n = \sum_{k=1}^n \mathbb{1}(X_k \ge u_n)$ and observe the equality of events $\{M^{(2)} < u_n\} = \{S_n < 2\}$.