Exercises

Extreme value theory

1. Show that $L(t) = \log t$ is a slowly varying function but t^ϵ is not if $\epsilon \neq 0$.

2. If the random variable X has distribution F with finite variance, then show that the condition
$$\lim_{x\to\infty} \frac{x^2 \mathbf{P}(|X| > x)}{\int_{|y|\le x} y^2 \mathrm{d}F(y)} = 0$$
automatically holds.

   Hint: To obtain a contradiction, suppose that $x^2\mathbf{P}(|X| > x)$ does not converge to 0, i.e. along a sequence x_n, it is at least ε. If this happens, then one can give an infinite lower bound on the second moment of X.

3. Let W_1 and W_2 be independent standard normal random variables.

   (a) Check that $1/W_2^2$ has the Lévy distribution, that is, the stable law with index $\alpha = 1/2$ and $\kappa = 1$, i.e. it has density
$$\frac{1}{\sqrt{2\pi y^3}} \exp\left(-\frac{1}{2y}\right) \quad \text{for } y \ge 0.$$

   (b) Prove that W_1/W_2 has Cauchy distribution.

4. Let X has a symmetric stable law with index α. Show that $\mathbf{E}|X|^p < \infty$ for $p < \alpha$.

   Hint: If $\varphi(t) = \mathbf{E}(e^{itX})$ denotes the characteristic function of X, then the following estimate can be used without proof for $u > 0$:
$$\mathbf{P}\left(|X| > \frac{2}{u}\right) \le \frac{1}{u}\int_{-u}^{u} (1 - \varphi(t))\mathrm{d}t$$
and the identity
$$\mathbf{E}(|X|^p) = \alpha \int_0^\infty t^{p-1}\mathbf{P}(|X| > t)\mathrm{d}t.$$

5. For $f : \mathbb{R} \to \mathbb{R}$ functions, consider Cauchy's functional equation
$$f(x + y) = f(x) + f(y).$$
Prove that the only solutions for the equation are the linear functions $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$, if we assume that f is

   (a) continuous;

   (b) continuous at one point;

(c) monotonic on any interval;

(d) bounded on any interval;

(e) measurable.

Hint: Luzin's theorem ensures that by the measurability of f there is a subset $F \subseteq [0,1]$ with Lebesgue measure $2/3$ such that f is uniformly continuous on F. Use this to conclude that the function f is bounded on an interval $(0, \delta)$ for some small $\delta > 0$.

6. Let $X_1, X_2, \ldots$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ which have the same distribution function $F(x)$, but we do not assume anything about their dependence structure. Let $M_n := \max_{1 \le i \le n} |X_i|$.

   (a) Suppose that there is an $\alpha > 0$ such that $\int_{-\infty}^{\infty} |x|^\alpha \mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(|X_i|^\alpha) < \infty$. Prove that for any $\varepsilon > 0$ and for fixed $\delta > 0$,
   $$\lim_{n \to \infty} \mathbf{P}\left(n^{-(1/\alpha + \varepsilon)} |M_n| > \delta\right) = 0.$$

   (b) Suppose that there is an $s > 0$ such that $\int_{-\infty}^{\infty} \exp(s|x|) \mathrm{d}F(x) < \infty$, that is, $\mathbf{E}(\exp(s|X_i|)) < \infty$. Prove that for any sequence b_n that goes to ∞ and for fixed $\delta > 0$,
   $$\lim_{n \to \infty} \mathbf{P}\left((b_n \log n)^{-1} |M_n| > \delta\right) = 0.$$

   Hint: Use the following inequality
   $$\mathbf{P}\left(\max_{1 \le i \le n} |X_i| > \lambda\right) = \mathbf{P}(\cup_{i=1}^n \{|X_i| > \lambda\}) \le \sum_{i=1}^n \mathbf{P}(|X_i| > \lambda) = n\mathbf{P}(|X_i| > \lambda)$$
   and a Markov type inequality.

7. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \to \infty} x^\alpha (1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim b x^{-\alpha}$ as $x \to \infty$), then show that the distribution of $(bn)^{-1/\alpha} M_n$ converges weakly to the Fréchet distribution:
   $$\mathbf{P}\left((bn)^{-1/\alpha} M_n < x\right) \to \mathbb{1}(x > 0) \exp\left(-x^{-\alpha}\right).$$

8. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x) < 1$ for all $x < \infty$ and $\lim_{x \to \infty} e^{\lambda x} (1 - F(x)) = b$ for some fixed constants $\lambda, b \in (0, \infty)$ (that is, $1 - F(x) \sim b e^{-\lambda x}$ as $x \to \infty$), then show that the distribution of $\lambda M_n - \log(bn)$ converges weakly to the Gumbel distribution:
   $$\mathbf{P}\left(\lambda M_n - \log(bn) < x\right) \to \exp\left(-e^{-x}\right).$$

9. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function $F(x) = \mathbf{P}(X_i < x)$. Let $M_n := \max_{1 \le i \le n} X_i$. If $F(x_0) = 1$ and $F(x) < 1$ for all $x < x_0$ and $\lim_{x \to x_0} (x_0 - x)^{-\alpha} (1 - F(x)) = b$ for some fixed constants $\alpha, b \in (0, \infty)$ (that is, $1 - F(x) \sim b(x_0 - x)^\alpha$ as $x \to x_0$),

then show that the distribution of $(bn)^{1/\alpha}(M_n - x_0)$ converges weakly to the Weibull distribution:

$$\mathbf{P}\left((bn)^{1/\alpha}(M_n - x_0) < x\right) \to \mathbb{1}(x < 0) \exp\left(-(-x)^\alpha\right).$$

10. Let $X_1, X_2, \ldots$ be an iid sequence of Poisson random variables with parameter $\lambda > 0$, that is,

$$\mathbf{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \ldots$ and let $M_n := \max_{1 \le i \le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converge to 0 for the integers $k \to \infty$.

11. Let $X_1, X_2, \ldots$ be an iid sequence of negative binomial random variables with parameters $p \in (0, 1)$ and $m \in \mathbb{N}$, that is,

$$\mathbf{P}(X = k) = \binom{k + m - 1}{k} p^m (1 - p)^k$$

for $k = 0, 1, 2, \ldots$ and let $M_n := \max_{1 \le i \le n} X_i$. Show that there is no such normalization under which the sequence of maxima M_n has a non-degenerate limit law.

Hint: Show that the necessary condition

$$\lim_{x \uparrow x_F} \frac{\overline{F}(x+)}{\overline{F}(x)} = 1$$

fails to hold along a sequence of integers where $\overline{F}(x) = 1 - F(x)$ is the tail probability function. More precisely $\overline{F}(k+1)/\overline{F}(k)$ converges to $1 - p$ for the integers $k \to \infty$. To this end, show and use the asymptotic identity

$$\mathbf{P}(X = k) = \frac{k^{m-1}}{(m-1)!} p^m (1 - p)^k (1 + o(1))$$

as $k \to \infty$.

12. Suppose that $X > 0$ is a random variable and $\alpha > 0$ is a number. Show that the following are equivalent:

    (a) X has the Fréchet distribution function

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$$

(b) $\ln X^\alpha$ has the Gumbel distribution function

$$\Lambda(x) = \exp(-e^{-x})$$

(c) $-X^{-1}$ has the Weibull distribution function

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$