Exercise-book
-
Part-III
Continuous distributions in one-dimension
-

Andras Vetier

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1 Continuous random variables

PROBLEMS

1. *Are they continuous?*
   Which of the following random variables are continuous?
   (a) the temperature measured in Celsius in the center of a city at noon time in summer;
   (b) the number of days when the temperature is higher than 30 Celsius in the center of a city at noon time during a summer;
   (c) the amount of time when, the temperature is higher than 30 Celsius in the center of a city during a day in summer;,
   (d) the number of students who attend a lecture at a university;
   (e) the weight (in kg) of the highest student among all students who attend a lecture at a university;
   (f) the total weight of all students who attend a lecture at a university;

2 Distribution function

EXCEL

The following file shows the graphs of the distribution functions of the most important continuous distributions.

Demonstration file: Distribution functions of the most important continuous distributions
ef-200-57-50-distr

PROBLEMS

2. *Calculating probabilities from the distribution function*
   Assume that the distribution function of a random variable X is $F(x) = \frac{x}{2}$ for $0 \leq x \leq 2$. How much is
   (a) $P(X < 1)$;
   (b) $P(X > 3)$;

(c) $P(X > 0.7)$;
(d) $P(0.3 < X < 0.7)$;
(e) $P(0.3 < X < 0.7|X > 1)$?

3. *Calculating probabilities from the distribution function*
Assume that the distribution function of a random variable X is $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$, or equivalently, $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$, for all x. How much is

   (a) $P(X < 1)$;
   (b) $P(X > 3)$;
   (c) $P(X < -1)$;
   (d) $P(X > -3)$;
   (e) $P(1 < X < 5)$;
   (f) $P(1 < X < 5|X > 3)$?

   Solution
   Sol-03-02-01

4. *Calculating probabilities from the distribution function*
Assume that the distribution function of a random variable X is $F(x) = 1 - e^{-x}$, if $x > 0$, and $f(x) = 0$ otherwise. How much is

   (a) $P(X < 1)$;
   (b) $P(X > 3)$;
   (c) $P(X < -1)$;
   (d) $P(X > -3)$;
   (e) $P(1 < X < 5)$;
   (f) $P(1 < X < 5|X > 3)$?

5. *Finding the distribution function*
A random number is generated by a calculator or computer. Let X be its

   (a) cube;
   (b) logarithm;
   (c) square;
   (d) square-root;
   (e) reciprocal.

   Find the distribution function of X in each case.

6. *Finding the distribution function*
Calculate $F(x)$ if

(a) $X = \mathrm{RND}^2$;

(b) $X = \mathrm{RND}^3$;

(c) $X = \sqrt{\mathrm{RND}}$;

(d) $X = 1/\mathrm{RND}$;

(e) $X = 1/\mathrm{RND}^2$;

(f) $X = -\ln(\mathrm{RND})$.

7. *Finding the distribution function*
Four random numbers are generated: RND_1, RND_2, RND_3, RND_4.

    (a) Let X_1 denote the largest value of the values RND_1, RND_2, RND_3,RND_4. Find the distribution function X_1.

    (b) Let X_2 denote the second largest value of the values RND_1, RND_2, RND_3,RND_4. Find the distribution function of X_2.

8. *Finding the distribution function*
Ten random numbers are generated by a calculator or computer. Let X be the 8th smallest of them. Find the distribution function of X.

9. *Constructing the graphs of distribution functions*
Make an Excel file to visualize the graphs of the distribution functions of the random variables given in the previous problems.

3 *** Empirical distribution function

EXCEL

The following file shows the construction of an empirical distribution function

Demonstration file: Empirical distribution function
ef-200-01-00

PROBLEMS

10. *Empirical distribution function of the maximum of two random numbers*
The following file is a small modification of the previous one. Some cells on page "1" are colored light yellow, some cells on page "s3" are colored dark yellow. Modify the light yellow cells: replace the power of the random number by the maximum of two random numbers. Write the Excel formula of the distribution function of the maximum of two random numbers into the dark yellow

cells, which is, x^2. You will see that the empirical distribution function will be oscillating around the distribution function.

Demonstration file: Empirical distribution function
eg-030-03-01

11. *Empirical distribution function of your own random variable*
Choose a distribution so that you can easily simulate it and its distribution function can be easily given by a formula like in the previous problem. Now use this distribution to do the steps of the previous problem.

4 Density function

EXCEL

The following file shows the graphs both of the distribution functions and of the density functions of the most important continuous distributions.

Demonstration file: Distribution functions of the most important continuous distributions
ef-200-57-50

PROBLEMS

12. *Finding the distribution function from the density function*
Assume that the density function of the random variable X is $f(x) = 1/8\ x$ if $0 < x < 2$. Find the distribution of X.

13. *Finding the distribution function from the density function*
Assume that the density function of the random variable X is $f(x) = 1/8\ x$ if $0 < x < 2$. Find the distribution function of $1/X$.

14. *Finding the distribution function from the density function*
Assume that the density function of X is $f(x) = 3\ x^2$ if $0 < x < 1$. Find the distribution function of X^6 .

15. *Calculating probabilities from the density function*
Assume that the density function of a random variable X is $f(x) = e^{-x}$, if $x > 0$, and $f(x) = 0$ otherwise. How much is
    (a) $P(X < 1)$;
    (b) $P(X > 3)$;

(c) $P(1 < X < 5)$;

(d) $P(1 < X < 5 | X > 3)$?

16. *Calculating probabilities from the density function*
Assume that the density function of a random variable X is $f(x) = \frac{1}{\pi\sqrt{1-x^2}}$ if $-1 < x < 1$. How much is

(a) $P(X < 0.5)$;

(b) $P(X > 0.3)$;

(c) $P(X > 0.7)$;

(d) $P(0.3 < X < 0.7)$;

(e) $P(0.3 < X < 0.7 | X > 0.5)$?

17. *Finding the value of the constant c*
Assume that the density function of the random variable X is $f(x) = c\,x^2 \;\; (0 < x < 3)$, where c is a constant . Find the value of c, and determine the distribution of X.

18. *Finding the density function*
A random number is generated by a calculator or computer. Let X be its

(a) cube;

(b) logarithm;

(c) square;

(d) square-root;

(e) reciprocal.

Find the density function of X in each case.

19. *Finding the density function*
Calculate $f(x)$ if

(a) $X = \text{RND}^2$;

(b) $X = \text{RND}^3$;

(c) $X = \sqrt{\text{RND}}$;

(d) $X = 1/\text{RND}$;

(e) $X = 1/\text{RND}^2$;

(f) $X = -\ln(\text{RND})$.

20. *Finding the density function*
Four random numbers are generated: RND_1, RND_2, RND_3, RND_4.

(a) Let X_1 denote the largest value of the values RND_1, RND_2, RND_3,RND_4. Find the density function X_1.

(b) Let X_2 denote the second largest value of the values RND_1, RND_2, RND_3,RND_4. Find the density function of X_2.

21. *Finding the density function*
Ten random numbers are generated by a calculator or computer. Let X be the 8th smallest of them. Find the density function of X.

22. *Constructing the graphs of density functions*
Make an Excel file to visualize the graphs of the density functions of the random variables given in the previous problems.

23. *"Linear" density function*
Make an Excel file to visualize the graph of the density function which is linear on the interval $[0; 1]$, that is, $f(x) = ax + b$ if $0 < x < 1$. Use b as a parameter. Pay attention to the fact that the total area under the density function must be equal to 1.

    Solution
    Sol-03-04-01

24. *"Linear" density function*
Make an Excel file to visualize the graph of the density function which is linear on the interval $[A; B]$, that is, $f(x) = ax + b$ if $A < x < B$. Use b as a parameter. Pay attention to the fact that the total area under the density function must be equal to 1.

5 *** Histogram

EXCEL

The following files show a histogram construction.

Demonstration file: Histogram construction
ef-200-04-00

Demonstration file: Histogram construction for standard normal distribution
ef-200-05-00

PROBLEM

25. *Histogram construction*
The following file is a small modification of the previous one. Some cells on page "1" are colored light yellow, some cells on page "s3" are colored dark yellow. Modify the light yellow cells so that you put there a random variable whose values are between -3 and 3. Write the Excel formula of the density function of this random variable into the dark yellow cells. You will see that the histogram will be oscillating around the density function.

Demonstration file: Histogram construction
eg-030-03-02

6 Uniform distributions

EXCEL

The following file shows the distribution functions and the density functions of uniform distributions, and gives also a simulation for this distribution.

Demonstration file: Uniform distribution: distribution function, density function, simulation
ef-200-07-00

PROBLEMS

26. *Calculating probabilities*
Assume that X follows uniform distribution on the interval $[0, 5]$. Calculate the probabilities and conditional probabilities:

    (a) $P(X < 3)$;
    (b) $P(1 < X)$;
    (c) $P(1 < X \text{ and } X < 3)$;
    (d) $P(1 < X \mid X < 3)$;
    (e) $P(X < 3 \mid 1 < X)$.

27. *Calculating probabilities*
Assume that X follows uniform distribution on the interval $[0.8, 12.3]$. Calculate the probabilities and conditional probabilities:

    (a) $P(X < 3.3)$;
    (b) $P(1.8 < X)$;

(c) $P(1.8 < X \text{ and } X < 3.3)$;

(d) $P(1.8 < X \mid X < 3.3)$;

(e) $P(X < 3.3 \mid 1.8 < X)$.

28. *Waiting time for the bus*
Assume that the waiting time for the bus follows uniform distribution between 0 and c minutes, but the constant c is not known. How much is c, if

(a) $P(\text{waiting time} < 5) = 0.25$;

(b) $P(\text{waiting time} < 5) = 0.5$;

(c) $P(\text{waiting time} < 5 \mid \text{waiting time} < c/3) = 0.5$.

7 Distributions of some functions of random numbers

EXCEL

The following files give simulations for random variables derived from uniformly distributed random numbers generated by a computer.

Demonstration file: Square of a random number, simulation and density function
ef-200-48-00

Demonstration file: Square root of a random number: simulation and density function
ef-200-50-10

Demonstration file: Random number to a positive power, simulation
ef-200-51-00

Demonstration file: Random number to a positive power times a constant, simulation
ef-200-51-50

Demonstration file: Reciprocal of a random number, simulation
ef-200-52-00

Demonstration file: A random number raised to a negative exponent, simulation
ef-200-53-00

Demonstration file: A random number raised to a negative exponent times a constant, simulation
ef-200-54-00

Demonstration file: Product of two random numbers, simulation
ef-200-55-00

Demonstration file: Ratio of two random numbers, simulation
ef-200-56-00

PROBLEMS

29. *Finding the distribution function and the density function*
Find the distribution function and the density function of the following random variables:

(a) $X = 5 * \text{RND}$;
(b) $X = 2 + 5 * \text{RND}$;
(c) $X = -2 + 5 * \text{RND}$;
(d) $X = \text{RND}^2$;
(e) $X = \text{RND}^{-2}$;
(f) $X = \text{RND}^3$;
(g) $X = \text{RND}^c$, where c is a positive constant;
(h) $X = \text{RND}^c$, where c is a negative constant;
(i) $X = 1/\text{RND}^2$;
(j) $X = \ln(\text{RND})$;
(k) $X = -\ln(\text{RND})$,
(l) $X = 25\sqrt{\text{RND}}$;
(m) $X = t(\text{RND})$, where $y = t(x)$ is a strictly increasing continuously differentiable function;
(n) $X = t(\text{RND})$, where $y = t(x)$ is a strictly decreasing continuously differentiable function.

30. *Finding the distribution function and the density function*
Two random numbers are generated: RND_1, RND_2. Find the distribution function and the density function of

(a) their maximum;
(b) RND_1RND_2;
(c) $\text{RND}_1\text{RND}_2^2$;

(d) $\mathrm{RND}_2/\mathrm{RND}_1$;

(e) $\mathrm{RND}_2/\mathrm{RND}_1^2$;

(f) $\mathrm{RND}_2^2/\mathrm{RND}_1^2$.

31. *Making simulations*
Make simulations of the random variables of the previous problems.

8 *** Arc-sine distribution

EXCEL

The following files show the arc-sine distribution.

Demonstration file: Arc-sine distribution
ef-200-09-00

Demonstration file: Arc-sine distribution
ef-200-10-00

PROBLEMS

32. *Constructing the graph of the density function*
Construct the graph of the density function of the arc-sine distribution.

33. *Arc-sine distribution on the interval $(-A; A)$*
Generalize the notion of the arc-sine distribution so that the $(-1; 1)$ interval is replaced by the interval $(-A; A)$. Set up the formula of the

    (a) distribution

    (b) density

    function of this generalized distribution. This distribution may be called arc-sine distribution with parameter A).

34. *Arc-sine distribution on the sky*
The moon of Jupiter called Io goes around Jupiter on a (practically) circular path. The Earth is in plane of this circle. If, using a telescope, you look at Io, then you may see it between the two extreme positions. If the time instant of your experiment is chosen at random, then the position of Io is a random point between the two extreme positions. Check that the position of IO follows the arc-sine distribution.

9 *** Cauchy distribution

EXCEL

The following files show the arc-sine distribution.

Demonstration file: Cauchy distribution
ef-200-11-00

Demonstration file: Cauchy distribution
ef-200-12-00

PROBLEMS

35. *Constructing the graph of the density function*
Construct the graph of the density function of the Cauchy distribution.

36. *Cauchy distribution with parameters A and B*
Assume that the random variable X follows the Cauchy distribution. Choose two constants, say A and B, and consider the random variable $Y = AX + B$. Find the formula of the distribution function and density function of Y. We may call the distribution of Y "the Cauchy distribution with parameters A and B".

10 *** Beta distributions

EXCEL

The following files show beta distributions.

Demonstration file: Beta distributions
ef-200-13-00

Demonstration file: Beta distributions
ef-200-14-00

Demonstration file: Beta distributions
ef-200-15-00

PROBLEMS

37. *Constructing the graph of the density function*
Construct the graphs of the density functions of the beta distributions related to

    (a) size 2 and rank 1;
    (b) size 2 and rank 2.

38. *Constructing the graph of the density function*
Construct the graphs of the density functions of the beta distributions related to

    (a) size 3 and rank 1;
    (b) size 3 and rank 2.

39. *Constructing the graph of the density function*
Construct the graphs of the density functions of the beta distributions related to

    (a) size 5 and rank 1;
    (b) size 5 and rank 2;

40. *Making simulations*
Make simulations of the random variables with the distributions of the previous problems.

41. *Calculating probabilities*
Three people, independently of each other, arrive to a party according to uniform distribution between 7pm and 7:30pm. What is the probability that

    (a) the first of them arrives
        i. between 7:10pm and 7:15pm?
        ii. between 7:15pm and 7:20pm?
    (b) the second of them arrives
        i. between 7:10pm and 7:15pm?
        ii. between 7:15pm and 7:20pm?
    (c) the third of them arrives
        i. between 7:10pm and 7:15pm?
        ii. between 7:15pm and 7:20pm?

42. *Making simulations*
Make simulations for the previous problems, and check that the relative frequency is always close to the probability.

11 Exponential distribution

EXCEL

The following files show beta distributions.

Demonstration file: Exponential distribution
ef-200-16-00

Demonstration file: Exponential distribution
ef-200-17-00

PROBLEMS

43. *Calculating probabilities*
X has an exponential distribution with parameter $1/3$. Find the probabilities:
    (a) $P(X < 3.5)$;
    (b) $P(X > 2.5)$;
    (c) $P(2.5 < X < 3.5)$;
    (d) $P(X < 3.5 \mid X > 2.5)$;
    (e) $P(X > 2.5 \mid X < 3.5)$.

44. *Making simulations*
Make simulations for the previous problem, and check that the (conditional) relative frequency is always close to the (conditional) probability.

45. *Finding a constant*
X has an exponential distribution with parameter $1/3$. Find the constant c so that $P(X < c) = 0.5$.

46. *Finding a constant*
X has an exponential distribution with parameter $1/3$. Find the constant c so that $P(X > c) = 0.7$.

47. *Life-time of an object with the memoryless property*
Assume that the life-time X of an object has the memoryless property, and thus it follows exponential distribution. Assume that $P(X > 4.5) = 0.3$. Find the probabilities:
    (a) $P(X < 1)$;
    (b) $P(X > 3)$;

(c) $P(1 < X < 5)$;

(d) $P(1 < X < 5 | X > 3)$?

48. *Life-time of a cup*
Let X be the life-time of a cup in a self-service restaurant (measured in months). Assume that the life-time follows an exponential distribution, and the average life-time is 3.5 months. Calculate the probabilities

(a) $P(3 < X)$;

(b) $P(13 < X | 10 < X)$.

49. *Earth-quakes*
The amount of time between two earth-quakes on an island follows exponential distribution with an average of 3 years.

(a) Assuming that 3 years have passed since the last earth-quake, what is the probability that there will be an earth-quake during the next year?

(b) Assuming that there will be an earth-quake during the next year, what is the probability that it will happen in the second half of the year?

50. *Electric bulbs*
The life-time of a certain type of electric bulbs has an exponential distribution with an average life time of 150 hours.

(a) What is the probability that a bulb of this type will work at least 200 hours?

(b) What is the probability that a bulb of this type having been used successfully for 100 hours will work at least 200 more hours in addition to the 100 hours?

51. *Life-time of a transistor*
Assume that the life time of a transistor has an exponential distribution with an average life time of 5 months.

(a) What is the probability that the transistor has a life time between 2 and 3 months?

(b) On condition that the transistor has a life time between 1 and 4 months, what is the probability that the transistor has a life time between 2 and 3 months?

52. *Two electrical components*
Suppose that two independent electrical components are exponentially distributed with a common average of 2.5 years. What is the probability that at least one of them has a life-time greater than 3 years?

53. *Calculating probabilities and expected values*
Assume that X is exponentially distributed with parameter 2.

(a) What is the probability that X is between 1 and 4 ?

(b) On condition that $X > 1$, what is the probability that X is between 1 and 4?

(c) On condition that $X < 4$, what is the probability that X is between 1 and 4?

(d) On condition that $X < 5$, what is the probability that X is between 1 and 4?

(e) Find the expected value of the distance between X and 3.

(f) Find the expected value of the distance between X and its expected value.

(g) Find c so that $P(X < c) = 0.25$.

(h) Find c so that $P(X > c) = 0.25$.

54. *Calculating probabilities and expected values*
Assume that X is exponentially distributed with parameter λ.

(a) What is the probability that X is between a and b ?

(b) On condition that $X > a$, what is the probability that X is between a and b?

(c) On condition that $X < b$, what is the probability that X is between a and b?

(d) On condition that $X < d$, what is the probability that X is between a and b?

(e) Find the expected value of the distance between X and its expected value.

(f) Find c so that $P(X < c) = p$.

(g) Find c so that $P(X > c) = q$.

55. *Calculating probabilities and expected values*
The life-time of an electrical component of a computer is assumed to satisfy the memoryless property. If its average life-time is 2.5 years, then what is the probability that such a component survives

(a) more than 1 year?

(b) more than 2 years?

(c) more than 3 years?

(d) more than x years?

56. *Average life-time of an electrical component*
The life-time of an electrical component of a computer is assumed to satisfy the memoryless property. Assume that the of probability that such a component survives more than 2.5 years is 0.7. How much is the average life-time of this electrical component?

57. *Reciprocal of the arrival time*
The random arrival time, denoted by X, of the goods I am waiting for has an exponential distribution with an expected value of 3 hours. My profit, denoted by Y, from the goods is proportional to the reciprocal of the arrival time: $Y = 10/X$. Determine the distribution function and the median of Y.

12 *** Gamma distribution

EXCEL

The following files show beta distributions.

Demonstration file: Gamma distribution
ef-200-25-00

Demonstration file: Gamma distribution
ef-200-26-00

Demonstration file: Gamma distribution
ef-200-19-00

Demonstration file: Gamma distribution
ef-200-28-00

Demonstration file: Gamma distribution
ef-200-29-00

Demonstration file: Gamma distribution
ef-200-30-00

Demonstration file: Gamma distribution
ef-200-31-00

Demonstration file: Gamma distribution
ef-200-32-00

PROBLEMS

58. *Calculating a probabilities*
X has a gamma distribution with parameter $n = 2$ and $\lambda = 1/3$. Find the probabilities:

(a) $P(X < 6.5)$;
(b) $P(X > 5.5)$;
(c) $P(5.5 < X < 6.5)$;
(d) $P(X < 6.5 \mid X > 5.5)$;

(e) $P(X > 5.5 \mid X < 6.5)$.

59. *Calculating a probabilities*
X has a gamma distribution with parameter $n = 3$ and $\lambda = 1/3$. Find the probabilities:

(a) $P(X < 9.5)$;

(b) $P(X > 8.5)$;

(c) $P(8.5 < X < 9.5)$;

(d) $P(X < 9.5 \mid X > 8.5)$;

(e) $P(X > 8.5 \mid X < 9.5)$.

60. *Making simulations*
Make simulations for the previous problem, and check that the (conditional) relative frequency is always close to the (conditional) probability.

13 Normal distributions

EXCEL

The following files show normal distributions.

Demonstration file: Normal distribution
ef-200-35-05

Demonstration file: Normal distribution
ef-200-33-00

Demonstration file: Normal distribution
ef-200-64-00

Demonstration file: Normal distribution
ef-200-35-00

Demonstration file: Normal distribution
ef-200-36-00

PROBLEMS

61. *Calculating probabilities*
Assume that X follows standard normal distribution. Calculate the following probabilities:

   (a) $P(X > 1.5)$;
   (b) $P(-1.3 < X < 2.5)$;
   (c) $P(-1.3 < X < 2.5 | X > 1.5)$;
   (d) $P(X > 1.5 | -1.3 < X < 2.5)$.

   Solution
   Sol-03-13-01

62. *Calculating probabilities*
Assume that X follows normal distribution with parameters $\mu = 220, \sigma = 10$. Calculate the following probabilities:

   (a) $P(X > 225)$;
   (b) $P(215 < X < 229)$;
   (c) $P(215 < X < 229 | X > 225)$;
   (d) $P(X > 225 | 215 < X < 229)$.

63. *Height of a man*
Assume that the height of a randomly chosen man in Hungary measured in cm is normally distributed with parameters $\mu = 180$ and $\sigma = 10$. What is the probability that the height of a randomly chosen man in Hungary is

   (a) less than 170?
   (b) greater than 185?
   (c) between 170 and 185?
   (d) smaller than 165 on condition that he is smaller than 185?
   (e) greater than 170 on condition that he is smaller than 185?

   Solution
   Sol-03-13-02

64. *Height of a man*
Assume that the height of a randomly chosen man in a country has a normal distribution. The expected value is 175 cm, the standard deviation is 15 cm.

   (a) What is the probability that a randomly chosen man is
      i. shorter than 150 cm?
      ii. taller than 200 cm?
      iii. has a height between 150 and 200 cm?

(b) Which is the height c so that

   i. the probability that a randomly chosen man is shorter than c is equal to 0.9?

   ii. the probability that a randomly chosen man is taller than c is equal to 0.8?

65. *Weight of a sack of potato*
Let us assume that the weight X (measured in kg) of a sack of potato follows normal distribution with parameters $\mu = 10$ and $\sigma = 0.1$. Sketch a nice graph of the density function of X. Calculate $P(9.9 < X < 10.1)$.

66. *Simulation of a standard normal random variable*
There are several ways to simulate a standard normal random variable. Check that each of the following simulations yields a standard normal random variable.

   (a) $X =$ `NORMSINV(`RND`);`

   (b) $X =$ `NORMINV(`RND`;0;1);`

   (c) $X = (\text{RND}_1 + \text{RND}_2 + ... + \text{RND}_{12}) - 6$,
   where $\text{RND}_1, \text{RND}_2, ..., \text{RND}_{12}$ are independent random variables, uniformly distributed between 0 and 1;

   (d) $X = \sqrt{(2\ln(\text{RND}_1))}\ \cos(\text{RND}_2)$,
   where $\text{RND}_1, \text{RND}_2$ are independent random variables, uniformly distributed between 0 and 1.

   Making several thousand simulations, you may be convinced that the last two simulations are much more efficient than the first two in the sense that they use much less time.

67. *Height of a woman*
Assume that the height of a randomly chosen woman in a country has a normal distribution. The expected value is 175 cm, the standard deviation is 15 cm.

   (a) What is the probability that a randomly chosen man is

      i. shorter than 150 cm?

      ii. taller than 200 cm?

      iii. has a height between 150 and 200 cm?

   (b) Which is the height c so that

      i. the probability that a randomly chosen man is shorter than c is equal to 0.9?

      ii. the probability that a randomly chosen man is taller than c is equal to 0.8?

68. *Height of teenager boys*
Let us assume that the height X of teenager boys (measured in cm-s) in a certain country follows normal distribution with parameters $\mu = 172,2$ and $\sigma = 12,6$.

Using Excel, make the graphs of the density function and of the distribution function. Calculate with Excel, for 4 decimal precision, the following unconditional and conditional probabilities :

(a) $P(X < 200)$;

(b) $P(X > 160)$;

(c) $P(165 < X < 185)$;

(d) $P(165 < X | X < 185)$;

(e) $P(165 < X < 185 | X < 190)$.

69. *Height of teenager boys in other countries*
The height X of teenager boys in other countries also follows normal distribution, but with other parameters. Let us assume that in European countries $\sigma = 12.6$, but μ may be different in different countries. For some reason, we want to know how the probabilities

(a) P(height is more than 160 cm);

(b) P(height is more than 160 cm, but less than 180 cm).

depend on μ. Construct the graphs of these functions with Excel.

Solution
Sol-03-13-03

70. *Height of teenager boys in Asian countries*
Let us assume that in Asian countries σ is less, say it is $\sigma = 6.3$. How the probabilities

(a) P(height is more than 160 cm);

(b) P(height is more than 160 cm, but less than 180 cm).

depend on μ now? Construct the graphs of these functions, and compare them to the previous ones.

71. *Heights of teenager boys and girls*
Let us assume that the height of teenager boys (measured in cm-s) in a certain country follows normal distribution with parameters $\mu = 182.2$ and $\sigma = 12.6$, and the height of teenager girls follows normal distribution with parameters $\mu = 162.2$ and $\sigma = 9.1$. Let us assume that 75 percent of students in a high school are girls, 25 percent are boys.

(a) Using Excel, sketch the graph of the density function and of the distribution function of the height of a randomly chosen girl.

(b) Using Excel, sketch the graph of the density function and of the distribution function of the height of a randomly chosen boy.

(c) P(height of a randomly chosen girl $< 175) =?$

(d) $P(\text{height of a randomly chosen boy} < 175) =?$

(e) A student is chosen at random. The height of this student is X. $P(X < 175) =?$

(f) $P(165 < X < 185) =?$

(g) $P(165 < X < 185 | X < 190) =?$

(h) On condition that $X < 160$, what is the probability that the student is a boy?

(i) On condition that $X > 180$, what is the probability that the student is a boy?

(j) On condition that $165 < X < 175$, what is the probability that the student is a boy?

(k) Find a formula for the distribution function of X.

(l) Find a formula for the density function of X.

(m) Using Excel, sketch the graph of the distribution function and of the density function.

72. *Temperature measured in Celsius degrees*
The temperature X (on May 1, at a certain place) measured in Celsius has a normal distribution with expected value 15.5 and standard deviation 4.5. Calculate the probabilities:

(a) $P(X < 10 \text{ or } X > 20)$;

(b) $P(X < 5 \text{ or } X > 25)$;

(c) $P(X < 10 \text{ or } X > 20 | X < 5 \text{ or } X > 25)$;

(d) Find a so that $P(X < a) = 0.25$;

(e) Find b so that $P(X < b) = 0.75$.

73. *Temperature measured also in Kelvin degrees*

(a) The temperature X (on May 1, at a certain place) measured in Celsius degrees has a normal distribution with expected value 15.5 and standard deviation 4.5. Calculate the probabilities:

i. $P(X < 10 \text{ or } X > 20)$;

ii. $P(X < 5 \text{ or } X > 25)$;

iii. $P(X < 10 \text{ or } X > 20 | X < 5 \text{ or } X > 25)$.

(b) The same temperature measured in Kelvin degrees is denoted by Z. The conversion formula between X and Z is $Z = X + 273.15$.
The same temperature measured in Réaumur degrees is denoted by W. The conversion formula between X and W is: $W = 8/10X$.
The same temperature measured in Fahrenheit is denoted by Y. The conversion formula between X and Y is: $Y = 9/5X + 32$.
Figure out

i. the expected value of Z, W and Y;
ii. the standard deviation of Z, W and Y;
iii. make nice figures for the density functions of X, Z, W and Y, as well. Figures made by a computer are appreciated.

74. *Comparing Excel functions*
Check that, in Excel, NORMSDIST(x) and NORMDIST$(x; 0; 1;$ TRUE$)$ mean the same thing.

75. *Comparing Excel functions*
Check that, in Excel, NORMSDIST$((x-\mu)/\sigma)$ and NORMDIST$(x; \mu; \sigma;$ TRUE$)$ mean the same thing.

76. *Amount of pollution*
Assume that the amount of pollution (measured in milligramms) in a liter of water of a lake follows normal distribution with expected value 6.5 and standard deviation 0.5. What is the probability that the average of

(a) 4 experimental results is in the interval $[6.4; 6.6]$?
(b) 25 experimental results is in the interval $[6.4; 6.6]$?

77. *Finding a critical value c*
Assume that a measurement result X has a normal distribution with expected value 220 and standard deviation 10.

(a) What is the probability that X is larger than 225 ?
(b) If we know that the measurement result X is larger than 210 , then what is the probability that$X$ is larger than 225 ?
(c) How much is the critical value c for which $P(X > c) = 0.9$?
(d) How much is the critical value c for which $P(X > c|X > 210) = 0.9$?

78. *How does the probability depend on μ?*
X has a normal distribution with an expected value μ and standard deviation 5. Using Excel, make a graph of the following probabilities for $150 < \mu < 200$:

(a) $P(X < 180)$;
(b) $P(X > 170)$;
(c) $P(170 < X < 180)$;
(d) $P(165 < X < 185)$.

Take then the value of the standard deviation as a parameter, and study how the graphs change when you change the parameter.

14 *** Distributions derived from normal

EXCEL

The following files show distributions derived from normal distributions.

Demonstration file: Log-normal distribution
ef-200-38-00

Demonstration file: Chi-square distribution, n=3
ef-200-39-00

Demonstration file: Chi-square distribution
ef-200-40-00

Demonstration file: Chi-distribution, n=3
ef-200-41-00

Demonstration file: Chi-distribution
ef-200-42-00

Demonstration file: Student-distribution (T-distribution) and random variable, n=3
ef-200-43-00

Demonstration file: Student-distribution (T-distribution) and random variable
ef-200-44-00

Demonstration file: F-distribution, m=3, n=4
ef-200-45-00

Demonstration file: F-distribution
ef-200-46-00

PROBLEMS

79. *Simulation*
Simulate random variables following the distributions derived from normal.

80. *Densities*
Construct the graphs of the density functions of the distributions derived from normal.

81. *Distribution functions*
Construct the graphs of the distribution functions of the distributions derived from normal.

15 ***Generating a random variable with a given continuous distribution

EXCEL

Study the following file, and notice that if we define the random variable as $X = \text{RND}^c$, that is, we plug RND into $x = y^c$, then the empirical distribution function approaches the function $y = x^{1/c}$, which is the inverse of the function $x = y^c$. This fact means that the theoretical distribution function of $X = \text{RND}^c$ is really the inverse of the function into which we plug RND.

Demonstration file: Empirical distribution function
ef-200-01-00

PROBLEMS

82. *Generating a random variable which follows a given continuous distribution*
Choose a continuous distribution so that the formula of its distribution function and the formula of the inverse of the distribution function is known for you. Define a random variable by plugging a random number into the inverse of the distribution function. Then, making more experiments, construct the empirical distribution function of the random variable, and be convinced that the empirical distribution function approaches the the given distribution function.

83. *Uniformly distributed random variable on the interval* $[0; 1]$
Simulate a uniformly distributed random variable on the interval $[0; 1]$.

84. *Uniformly distributed random variable on* $[0; B]$
Simulate a uniformly distributed random variable on the interval $[0; B]$.

85. *Uniformly distributed random variable on* $[A; B]$
Simulate a uniformly distributed random variable on the interval $[A; B]$.

86. *"Linear" density function*
Assume that the density function of a random variable is linear on the interval $[0; 1]$, that is, $f(x) = ax + b$ if $0 < x < 1$. Use b as a parameter.

    (a) Calculate the distribution function.

    (b) Determine the inverse of the distribution function.

    (c) Make a point-cloud for a random variable which follows the distribution defined by such a density function.

Solution
Sol-03-15-01

87. *Density linearly increasing on* $[0; 1]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = cx$, if $0 < x < 1$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

88. *Density linearly increasing on the interval* $[0; B]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = cx$, if $0 < x < B$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

89. *Density linearly decreasing on* $[0; 1]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = c(1-x)$, if $0 < x < 1$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

90. *Density linearly decreasing on* $[0; B]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = c(B-x)$, if $0 < x < B$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

91. *Density quadratically increasing on* $[0; 1]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = cx^2$, if $0 < x < 1$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

92. *Density quadratically increasing on* $[0; B]$
Assume that the density function of a continuous distribution is linearly increasing on the interval $[0; 1]$, that is, $f(x) = cx^2$, if $0 < x < B$, and 0 otherwise, where c is a constant. How much is c? Determine the distribution function $F(x)$. Find the inverse of the distribution function. Use the inverse of the distribution function to simulate a random variable X which follows this distribution.

93. *Generating a random variable which follows a normal distribution*
Generate a random variable which follows

    (a) the normal distribution with $\mu = 75$ and $\sigma = 5$;

    (b) the normal distribution with $\mu = 175$ and $\sigma = 5$;

    (c) the normal distribution with $\mu = 175$ and $\sigma = 25$;

(d) the normal distribution with μ and σ, where μ and σ are parameters.

94. *Generating a random variable which follows an exponential distribution*
Generate a random variable which follows

(a) the exponential distribution with $\lambda = 2.5$;

(b) the exponential distribution with $\lambda = 5$;

(c) the exponential distribution with an expected value 2.

(d) the exponential distribution with λ, where λ is parameter.

16 Expected value of continuous distributions

EXCEL

The following file shows the expected values of some continuous distributions.

Demonstration file: Expected value of continuous distributions
ef-200-57-61

PROBLEMS

95. *Calculating the expected value from the density function*
Assume that the density function of a random variable X is

(a) $f(x) = 2x$, if $0 \leq x \leq 1$;

(b) $f(x) = x/2$, if $0 \leq x \leq 2$;

(c) $f(x) = \frac{1}{2\sqrt{x}}$, if $0 \leq x \leq 1$.

How much is the Expected value of X.

96. *Calculating the expected value from the distribution function*
Assume that the distribution function of a random variable X is

(a) $F(x) = \frac{x}{2}$, if $0 \leq x \leq 2$;

(b) $F(x) = 1 - e^{-x}$, if $x > 0$, and $f(x) = 0$ otherwise.

How much is the Expected value of X.

97. *Calculating the expected value of random variables*
A random number is generated by a calculator or computer. Let X be its

(a) cube;

(b) logarithm;
(c) square;
(d) square-root;
(e) reciprocal.

Find the expected value of X in each case.

98. *Calculating the expected value of random variables*
Calculate the expected value of X, if

(a) $X = \text{RND}^2$;
(b) $X = \text{RND}^3$;
(c) $X = \sqrt{\text{RND}}$;
(d) $X = 1/\text{RND}$;
(e) $X = 1/\text{RND}^2$;
(f) $X = -\ln(\text{RND})$.

99. *Calculating the expected value of random variables*
Four random numbers are generated: RND_1, RND_2, RND_3, RND_4.

(a) Let X_1 denote the largest value of the values RND_1, RND_2, RND_3,RND_4. Find the expected value of X_1.
(b) Let X_2 denote the second largest value of the values RND_1, RND_2, RND_3,RND_4. Find the expected value of X_2.

100. *Calculating the expected value of random variables*
Ten random numbers are generated by a calculator or computer. Let X be the 8th smallest of them. Find the expected value of X.

17 Expected value of a function of a continuous random variable

EXCEL

The following file shows the expected values of some functions of a random number.

Demonstration file: Expected value of some functions of a random number
ef-200-58-00

PROBLEMS

101. *Calculating the expected value of a function of a random variable*
The density function of X is $f(x) = 5x^4 \;\; (0 < x < 1)$. Let $Y = X^2$.

(a) Calculate the expected value of Y without using the density function of Y, but using the the density function of X.

(b) Calculate the distribution function and the density function of Y, and then the expected value of Y using the density function of Y.

102. *Calculating the expected value of a function of a random variable*
Calculate the expected value of

(a) $X =$ RND;

(b) $X =$ RND^2;

(c) $X =$ RND^3;

(d) $X = \sqrt{(\text{RND})}$.

Make simulation (1000 experiments) for each of the random variables above, and check that the average of the experimental result is close to the expected value.

103. *Calculating the expected value of a distance*
X has an exponential distribution with parameter $1/3$. Let us consider the distance

(a) between the random value X and the value 1, which is $|X - 1|$.

(b) between the random value X and the value 2, which is $|X - 2|$.

(c) between the random value X and a constant c, which is $|X - c|$.

Calculate the expected value of this distance.

104. *Calculating the expected value of a squared distance*
X has an exponential distribution with parameter $1/3$.

(a) Let us consider the squared distance between the random value X and the value 1, which is $(X - 1)^2$. Calculate the expected value of $(X - 1)^2$.

(b) Let us consider the squared distance between the random value X and the value 2, which is $(X - 1)^2$. Calculate the expected value of $(X - 2)^2$.

(c) Let us consider the squared distance between the random value X and a constant c, which is $(X - c)^2$. Calculate the expected value of $(X - c)^2$.

105. *Calculating the expected value of a distance*
Calculate the expected value of the distance from the expected value for the random variables:

(a) $X = 5 *$ RND;

(b) $X = 10 *$ RND;

(c) $X =$ RND^3.

106. *Calculating the expected value of a squared distance*
Calculate the expected value of the squared distance from the expected value for the random variables:

(a) $X = 5 * \text{RND}$;

(b) $X = 10 * \text{RND}$;

(c) $X = \text{RND}^3$.

107. *Approximating the average by a constant*
100 independent random numbers are generated between 0 and 1 according to uniform distribution. Determine a constant number which can be considered as a good approximation for the average of their cubes.

18 *** Median

EXCEL

The following file shows the median of the exponential distribution.

Demonstration file: Median of the exponential distribution
ef-200-57-00

PROBLEMS

108. *Calculating the median*
Calculate the median of the distribution with the distribution function: $F(x) = 2x(0 < x < 1)$. Compare it to the expected value.

109. *Calculating the median*
Calculate the median of the distribution with the distribution function: $F(x) = 2 - 2x(0 < x < 1)$. Compare it to the expected value.

110. *Calculating the median*
Calculate the median of the exponential distribution with parameter λ. Compare it to the expected value.

19 Standard deviation, etc.

EXCEL

The following files show the meaning of the standard deviation.

Demonstration file: Standard deviation for a data-set
ef-200-59-00

Demonstration file: Standard deviation for a random variable
ef-200-60-00

The following file shows the standard deviation of some distributions.

Demonstration file: Standard deviation for a data-set
ef-200-57-60

PROBLEMS

111. *Comparing the empirical variance to the theoretical*
We have learnt that the variance of the uniform distribution on the [0;1] interval is 1/12. Generate 1000 experimental results for the random variable

$$\text{RND}$$

and using the VARP command, be convinced that the variance of the generated data set is approximately 1/12.

112. *Comparing the empirical variance to the theoretical*
Generate 1000 experimental results for the random variable

$$\text{RND}_1 + \text{RND}_2$$

and using the VARP command, be convinced that the variance of the generated data set is approximately 2/12.

113. *Comparing the empirical variance to the theoretical*
Generate 1000 experimental results for the random variable

$$\text{RND}_1 + \text{RND}_2 + \text{RND}_3$$

and using the VARP command, be convinced that the variance of the generated data set is approximately 3/12.

114. *Comparing the empirical variance to the theoretical*
Generate 1000 experimental results for the random variable

$$\text{RND}_1 + \text{RND}_2 + ... + \text{RND}_{12}$$

and using the VARP command, be convinced that the variance of the generated data set is approximately 1.

115. *Comparing the empirical variance to the theoretical*
We have learnt that the variance of the standard normal distribution is 1. Generate 1000 random numbers by the NORMSINV(RAND()) or by the NORMINV(RAND();0;1) command, and using the VARP command, be convinced that the variance of the generated data set is approximately 1.

116. *Comparing the empirical variance to the theoretical*
Generate 1000 random numbers by the NORMINV(RAND();10;3) command, and using the VARP command. Be convinced that the variance of the generated data set is approximately 9.

117. *Comparing the empirical variance to the theoretical*
Generate 1000 random numbers by the NORMINV(RAND();10;4) command, and using the VARP command. Be convinced that the variance of the generated data set is approximately 16.

118. *Summation rule for the variance*
Let us take now the sum of the independent random variables generated in the previous 2 problems, that is, let us generate 1000 random numbers by the NORMINV(RAND();10;3) + NORMINV(RAND();20;4) command. Be convinced that the variance of the generated data set is approximately 25.

119. *Average rule for the variance*
X has a normal distribution with parameters $\mu = 300$ and $\sigma = 50$.

    (a) Calculate the probability that X is between 285 and 315.

    (b) We make 25 experiments for X. Calculate the probability that the average of the experimental results is between 285 and 315.

120. *25 sacks of potato*
Let us assume that the weight X (measured in kg-s) of a sack of potato follows normal distribution with parameters $\mu = 10$ and $\sigma = 0.1$. Assume that 25 sacks are examined. Let $\overline{X}_{25}$ be their average weight. Sketch a nice graph of the density function of $\overline{X}_{25}$. Calculate $P(9.9 < \overline{X}_{25} < 10.1)$.

20 *** Poisson-processes

EXCEL

The following files simulate Poisson processes with different intensity functions. First, second and third occurrences are observed in them.

Demonstration file: First (second and third) occurrence, homogeneous Poisson-process
ef-200-21-00

Demonstration file: First (second and third) occurrence, "trapezoid shaped" intensity function
ef-200-22-00

Demonstration file: First (second and third) occurrence, linearly increasing intensity function
ef-200-23-00

Demonstration file: First (second and third) occurrence, decreasing intensity function
ef-200-24-00

PROBLEMS

121. *Number of telephone calls during an hour*
Assume that during the working hours the average number of telephone calls arriving to our department in an hour is 4.5. What is the probability that, between 11 and 12am,

  (a) no calls arrive?
  (b) 1 call arrives?
  (c) 2 calls arrives?
  (d) less than 5 calls arrive?

122. *Number of telephone calls during 20 minutes*
Assume that during the working hours the average number of telephone calls arriving to our department in an hour is 4.5. What is the probability that, between 11 and 11:20am,

  (a) no calls arrive?
  (b) 1 call arrives?
  (c) 2 calls arrives?
  (d) less than 5 calls arrive?

123. *Number of telephone calls on condition that ...*
Assume that during the working hours the average number of telephone calls arriving to our department in an hour is 4.5. On condition that less than 5 calls arrive between 11 and 12am, what is the probability that 2 calls arrive

(a) between 11 and 12am?

(b) between 11 and 11:20am?

21 *** Transformation from line to line

EXCEL

Here are some files to study transformations from line to line.

Demonstration file: Uniform distribution transformed by a power function with a positive exponent
ef-200-95-00

Demonstration file: Uniform distribution transformed by a power function with a negative exponent
ef-200-96-00

Demonstration file: Uniform distribution on (A;B) transformed by a power function
ef-300-01-00

Demonstration file: Exponential distribution transformed by a power function with a positive exponent
ef-200-97-00

Demonstration file: Exponential distribution transformed by a power function with a negative exponent
ef-200-98-00

Demonstration file: Exponential distribution transformed by a power function
ef-200-99-00

PROBLEMS

124. *Transformation from line to line*
The density function of X is $f(x) = 5x^4 \ \ (0 < x < 1)$. Let $Y = X^2$. Calculate and visualize the distribution function and the density function of Y. Calculate the expected value, the variance and standard deviation of Y without using the density function of Y, but using the the density function of X. Calculate the expected value, the variance and standard deviation of Y using the density function of Y.

125. *Salt crystals*
The salt crystals have a cubic shape so that their size is random. Assume that the length X of a side of a salt crystal cube (measured in mm-s) has an exponential distribution with an expected value 0.5.

   (a) Calculate the expected value of the surface Y of a salt crystal ($Y = 6{*}X^2$).

   (b) Determine the formula of the distribution function $G(y) = P(Y < y) = ...$ and the formula of the density function $g(y)$ of the surface of a randomly chosen salt crystal.

   (c) Determine the formula of the distribution function $H(v) = P(V < v) =$ ... and the formula of the density function h(v) of the volume V of a randomly chosen a salt crystal ($V = X^3$).

   (d) Which is the critical volume-value for which the half of the salt crystals have a smaller volume than the critical value, and half of the salt crystals have a larger volume than the critical value?

126. *Log-normal distribution*
Assume that a measurement result X has a normal distribution. Determine the density function of the random variable $Y = exp(X)$. The distribution of Y is called: log-normal distribution, since the logarithm of Y has a normal distribution. Try to figure out: for what kind of real life problems we can use log-normal distribution.