Second Repeated Midterm Test (2016 12 07) – SOLUTIONS

1. The temperature at noon in January on a little island follows normal distribution with an expected value 7 °C and standard deviation 4 °C.

   (a) *(2 points)* What is the probability that the temperature is above 10 °C ?

   Solution:

$$\mathrm{P}(X > 10) = 1 - \Phi\left(\frac{10-7}{4}\right) = 1 - \Phi(0.75) = 0.225$$

   (b) *(3 points)* What is the probability that the temperature is above 10 °C on condition that it is above the average temperature?

$$\mathrm{P}(X > 10 \,|\, X > 7) = \frac{\mathrm{P}(X > 10)}{\mathrm{P}(X > 7)} = \frac{0.225}{0.5} = 0.45$$

2. The temperature at noon in January on a little island follows normal distribution with an expected value 7 °C and standard deviation 4 °C. The temperature at midnight follows normal distribution with an expected value 5 °C and standard deviation 2 °C. The correlation coefficient is 0.6.

   (a) *(2 points)* What is the expected temperature at noon if the midnight temperature is 4 °C ?

   Solution: X =temperature at noon, Y =temperature at midnight. Parameters:

$$\mu_1 = 7, \quad \sigma_1 = 4, \qquad \mu_2 = 5, \quad \sigma_2 = 2, \qquad r = 0.6$$

$$\mathrm{E}(X \,|\, Y = y) = \mu_1 + r \cdot \frac{\sigma_1}{\sigma_2} \cdot (y - \mu_2)$$

$$\mathrm{E}(X \,|\, Y = 4) = 7 + 0.6 \cdot \frac{4}{2} \cdot (4 - 5)\,) = 7 - 1.2 = 5.8$$

   (b) *(3 points)* What is the probability that temperature at noon is above 10 °C if the midnight temperature is 4 °C ?

   Solution:

$$\mathrm{SD}(X \,|\, Y = y) = \sigma_1 \cdot \sqrt{1 - r^2}$$

$$\mathrm{SD}(X \,|\, Y = 4) = 4 \cdot \sqrt{1 - 0.6^2} = 4 \cdot \sqrt{1 - 0.36} = 4 \cdot \sqrt{0.64} = 4 \cdot 0.8 = 3.2$$

$$\mathrm{P}(X > 10 \,|\, Y = 4) = 1 - \Phi\left(\frac{0 - 5.8}{3.2}\right) = 1 - \Phi(1.31) = 0.09$$

3. X follows the distribution which has a distribution function $F_1(x) = 1 - \frac{1}{x^4} \quad (x > 1)$. If $X = x$, then Y follows uniform distribution between 0 and $\frac{1}{x}$.

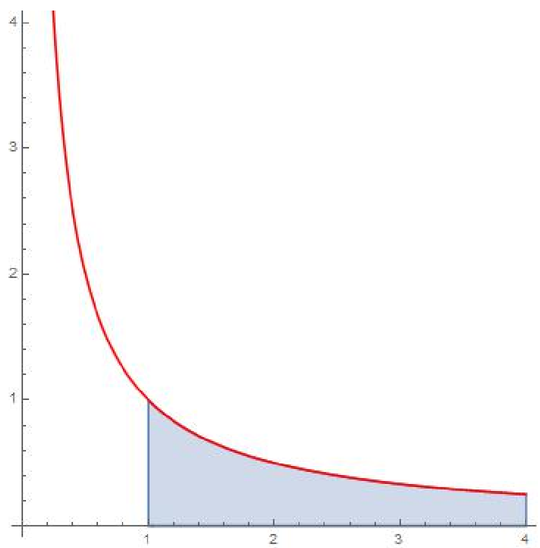
   (a) *(1 point)* Determine the density function of X.

   Solution:

$$f_1(x) = F_1'(x) = 4x^{-5} \;\; (x > 1)$$

   (b) *(1 point)* Draw the set of the possible values of (X, Y) .

$$\{(x, y) : \; x > 1, \quad 0 < y < \frac{1}{x}\}$$



(c) *(2 points)* Determine the density function of (X, Y).

$$f_{2|1}(y|x) = \frac{1}{\left(\frac{1}{x}\right)} = x \qquad (0 < y < \frac{1}{x})$$

$$f(x,y) = f_1(x) \cdot f_{2|1}(y|x) = 4x^{-5} \cdot x = 4x^{-4} \qquad (x > 1 \qquad 0 < y < \frac{1}{x})$$

(d) *(1 point)* Determine the density function of Y.

Solution:

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{1}^{\frac{1}{y}} 4x^{-4}\,dx = 4\left[\frac{x^{-3}}{-3}\right]_1^{\frac{1}{y}} = \frac{4}{3} - \frac{4}{3}y \qquad (0 < y < 1)$$

4. *(3+2 or 2+3 points)* Give the meaning of the the standard deviation of a continuous random variable

   (a) by a correct(!) mathematical formula,

$$\sqrt{\int_{-\infty}^{\infty} (\ x - \mu\ )^2 f(x)\,dx} \qquad \text{where} \qquad \mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

   or

$$\mathrm{SD}(X) = \sqrt{\mathrm{E}\left(\left[\ X - \mathrm{E}(X)\ \right]^2\right)}$$

   (b) in words, based on experimental results.

   Solution: If we make a large number of experiments, then the standard deviation of the experimental results is close to the theoretical standard deviation,