**First Repeated Midterm Test (2016 12 07) – SOLUTIONS**

1. Assume that, in a country, 0.02 part of cats are infected by a certain illness, and 0.98 part of them are healthy. Assume also that if a cat is infected by the illness, then it will be correctly diagnosed sick with a probability 0.85, and it will be mistakenly diagnosed healthy with a probability 0.15. Moreover, if a cat is healthy, then it will be correctly diagnosed healthy with a probability 0.75, and it will be mistakenly diagnosed sick with a probability 0.25,

   (a) *(3 points)* A cat is examined, and the test says the cat is sick. Knowing this fact what is the probability that this cat is really sick?
   **Solution:**

$$\text{P( sick } | \text{ diagnosed sick )} = \frac{\text{P( sick AND diagnosed sick )}}{\text{P( diagnosed sick )}} =$$

$$= \frac{\text{P( sick AND diagnosed sick )}}{\text{P( sick AND diagnosed sick )} + \text{P( healthy AND diagnosed sick )}} =$$

$$= \frac{\text{P( sick) P( diagnosed sick } \mid \text{ sick )}}{\text{P( sick)} \cdot \text{P( diagnosed sick } \mid \text{ sick )} + \text{P( healthy)} \cdot \text{P( diagnosed sick } \mid \text{ healthy )}} =$$

$$= \frac{(0.02)\ (0.85)}{(0.02)\ (0.85)\ +\ (0.98)\ (0.25)} \qquad (= 0.065)$$

   (b) *(2 points)* Now imagine that a cat is examined twice. The two examinations are independent. Both tests say the cat is sick. Knowing this fact what is the probability that this cat is really sick?
   **Solution:**

$$\text{P( sick } | \text{ twice diagnosed sick )} = \frac{\text{P( sick AND twice diagnosed sick )}}{\text{P( twice diagnosed sick )}} =$$

$$= \frac{\text{P( sick AND twice diagnosed sick )}}{\text{P( sick AND twice diagnosed sick )} + \text{P( healthy AND twice diagnosed sick )}} =$$

$$= \frac{\text{P(sick)P( twice diagnosed sick} \mid \text{sick)}}{\text{P( sick)} \cdot \text{P( twice diagnosed sick} \mid \text{sick)} + \text{P(healthy)} \cdot \text{P( twice diagnosed sick} \mid \text{ healthy)}} =$$

$$= \frac{(0.02)\ (0.85)^2}{(0.02)\ (0.85)^2\ +\ (0.98)\ (0.25)^2} \qquad (= 0.19)$$

2. *(5 points)* Assume that there are 250 seats on an air-plain, and 255 tickets are sold. Let us assume that each passenger may miss the flight independently of the others with a probability $p = 0.02$. How much is the probability that at least 2 persons will not have a seat on the air-plain?

   **Solution:**

$$\text{P( more than 251 passengers come )} = \sum_{k=252}^{255} \text{P( } k \text{ passengers come )} =$$

$$= \sum_{k=252}^{255} \binom{255}{k} (0.98)^k (0.02)^{255-k} \qquad (= 0.0.248)$$

3. $X$ has an exponential distribution with expected value 5. Find the probabilities:

(a) *(1 point)* $P(X < 3)$
**Solution:**

$$\mathrm{P}(X < 3) = F(3) = 1 - \mathrm{e}^{(-\frac{1}{5}\cdot 3)} = 1 - \mathrm{e}^{(-\frac{3}{5})} \qquad (= 0.451)$$

(b) *(2 points)* $P(X < 3 \mid X > 2)$
**Solution:**

$$\mathrm{P}(X < 3 \mid X > 2) = \frac{\mathrm{P}(X < 3 \text{ AND } X > 2)}{\mathrm{P}(X > 2)} = \frac{\mathrm{P}(2 < X < 3)}{\mathrm{P}(X > 2)} =$$

$$= \frac{F(3) - F(2)}{1 - F(2)} = \frac{\left(1 - \mathrm{e}^{(-\frac{1}{5}\cdot 3)}\right) - \left(1 - \mathrm{e}^{(-\frac{1}{5}\cdot 2)}\right)}{1 - \left(1 - \mathrm{e}^{(-\frac{1}{5}\cdot 2)}\right)} = \frac{\mathrm{e}^{(-\frac{1}{5}\cdot 2)} - \mathrm{e}^{(-\frac{1}{5}\cdot 3)}}{\mathrm{e}^{(-\frac{1}{5}\cdot 2)}} \qquad (= 0.181)$$

(c) *(2 points)* $P(X < 2 \mid X < 3)$
**Solution:**

$$\mathrm{P}(X < 2 \mid X < 3) = \frac{\mathrm{P}(X < 2)}{\mathrm{P}(X < 3)} = \frac{1 - \mathrm{e}^{(-\frac{1}{5}\cdot 2)}}{1 - \mathrm{e}^{(-\frac{1}{5}\cdot 3)}} = \qquad (= 0.731)$$

4. *(3+2 or 2+3 points)* Give the meaning of the the expected value of a discrete random variable

(a) by a correct(!) mathematical formula,
**Solution:**

$$\sum_x x\, p(x)$$

(b) in words, based on experimental results.
**Solution:** If the number of experiments is large, then the average of the experimental results is close to the expected value.