**Second Midterm Test (2016 11 26) – SOLUTIONS**

1. The temperature at noon in early December on a norwegian island follows normal distribution with an expected value $4.5$ °C and standard deviation $2$ °C.

   **(a)** *(2 points)* What is the probability that temperature is below $0$ °C ?

   **Solution:**
$$\mathrm{P}(X<0)=\Phi\left(\frac{0-4.5}{2}\right)=\Phi(-2.25)=1-\Phi(2.25)=1-0.99=0.01$$

   **(b)** *(3 points)* What is the probability that temperature is below $0$ °C on condition that it is below the average temperature?

   **Solution:**
$$\mathrm{P}(X<0\,|\,X<4.5)=\frac{\mathrm{P}(X<0)}{\mathrm{P}(X<4.5)}=\frac{0.01}{0.5}=0.02$$

2. The temperature at noon in early December on a norwegian island follows normal distribution with an expected value $4.5$ °C and standard deviation $2$ °C. The temperature at midnight follows normal distribution with an expected value $-1.5$ °C and standard deviation $1$ °C. The correlation coefficient is $0.8$.

   **(a)** *(2 points)* What is the expected temperature at noon if the midnight temperature is $-2.5$ °C ?

   **Solution:** $X$ =temperature at noon, $Y$ =temperature at midnight. Parameters:
$$\mu_1=4.5,\quad \sigma_1=2,\qquad \mu_2=-1.5,\quad \sigma_2=1,\qquad r=0.8$$
$$\mathrm{E}(X\,|\,Y=y)=\mu_1+r\cdot\frac{\sigma_1}{\sigma_2}\cdot(y-\mu_2)$$
$$\mathrm{E}(X\,|\,Y=-2.5)=4.5+0.8\cdot\frac{2}{1}\cdot(-2.5-(-1.5)\,)=4.5-1.6=2.9$$

   **(b)** *(3 points)* What is the probability that temperature at noon is below $0$ °C if the midnight temperature is $-2.5$ °C ?

   **Solution:**
$$\mathrm{SD}(X\,|\,Y=y)=\sigma_1\cdot\sqrt{1-r^2}$$
$$\mathrm{SD}(X\,|\,Y=-2.5)=2\cdot\sqrt{1-0.8^2}=2\cdot\sqrt{1-0.64}=2\cdot\sqrt{0.36}=2\cdot 0.6=1.2$$
$$\mathrm{P}(X<0\,|\,Y=-2.5)=\Phi\left(\frac{0-2.9}{1.2}\right)=\Phi(-2.42)=1-\Phi(2.42)=1-0.99=0.01$$

3. $X$ follows the distribution which has a distribution function $F_1(x)=x^4\ \ (0<x<1)$. If $X=x$, then $Y$ follows uniform distribution between 0 and $x^2$.

   **(a)** *(1 point)* Determine the density function of $X$.

   **Solution:**
$$f_1(x)=F_1'(x)=4x^3\ \ (0<x<1)$$

   **(b)** *(1 point)* Draw the set of the possible values of $(X,Y)$ .

**Solution:** The set of the possible values of $(X, Y)$ is the region with the boundary consisting of the following three parts:

$$\{(x,y):\ 0<x<1, \quad y=0\}$$
$$\{(x,y):\quad 0<x<1,\ y=x^2\}$$
$$\{(x,y):\ x=1, \quad 0<y<1\}$$

**(c)** *(2 points)* Determine the density function of $(X, Y)$.

**Solution:**

$$f_{2|1}(y|x) = \frac{1}{x^2} \qquad (0<y<x^2)$$
$$f(x,y) = f_1(x)\cdot f_{2|1}(y|x) = 4x^3\cdot\frac{1}{x^2} = 4x \qquad (0<x<1, \qquad 0<y<x^2)$$

**(d)** *(1 point)* Determine the density function of $Y$.

**Solution:**

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{\sqrt{y}}^{1} 4x\,dx = \left[x^2\right]_{\sqrt{y}}^{1} = 2-2y \qquad (0<y<1)$$

4. *(3+2 or 2+3 points)* Assume that the value of the distribution function of a random variable $X$ at $4.5$ is $0.3$.

**(a)** Explain briefly – in words – the meaning of this fact in terms of the notion of probability.

**Solution:** It means that the probability of the event that the random variable is less than $4.5$ is equal to $0.3$.

**(b)** Assume that 1000 experiments are made for the random variable $X$. What does the above fact mean for the experimental results?

**Solution:** It means that approximately 300 experimental results are less than $4.5$, and approximately 700 experimental results are greater than $4.5$.