Exam 2017 01 09 , ***SOLUTIONS***

1. In a box there are 5 red and 3 blue balls. You pick balls, one after the other without replacement. You stop picking when two red has already been drawn. Let X mean the number of draws.

   (a) Set up a table for the distribution of X.

   Solution: X can take on values 2,3,4,5. The probability mass function of X is (r=red, b=blue):

$$p(2) = \mathbb{P}(rr) = \frac{5}{8}\frac{4}{7} = \frac{5}{14}$$
$$p(3) = \mathbb{P}(rbr) + \mathbb{P}(brr) = \left(\frac{5}{8}\frac{3}{7} + \frac{3}{8}\frac{5}{7}\right)\frac{4}{6} = 2\cdot\frac{5\cdot 3}{8\cdot 7}\cdot\frac{2}{3} = \frac{5}{14}$$
$$p(4) = \mathbb{P}(rbbr) + \mathbb{P}(brbr) + \mathbb{P}(bbrr) = 3\cdot\frac{5\cdot 3\cdot 2}{8\cdot 7\cdot 6}\cdot\frac{4}{5} = \frac{3}{14}$$
$$p(5) = \mathbb{P}(rbbbr) + \mathbb{P}(brbbr) + \mathbb{P}(bbrbr) + \mathbb{P}(bbbrr) = 4\cdot\frac{5\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5}\cdot\frac{4}{4} = \frac{1}{14}.$$

   (b) If you made 1000 experiments, approximately how much would be the average of the number of draws?

   Solution: Since 1000 is „large”, the average of the number of draws is apprioximately the expectation

$$\mathbb{E}(X) = 2\cdot\frac{5}{14} + 3\cdot\frac{5}{14} + 4\cdot\frac{3}{14} + 5\cdot\frac{1}{14} = 3.$$

   (5 points for each part)

2. The life-time of an object has a uniform distribution between 0 and B. Approximately only 25 % of such objects live more than 6 years.

   (a) How much is the expected value of these objects?

   Solution: $f(x) = \frac{1}{B}$ if $0 < x < B$. So

$$\mathbb{P}(X > 6) = \int_6^B \frac{1}{B}dx = \frac{B-6}{B} = 0.25 = \frac{1}{4} \Longrightarrow B = 8 \Longrightarrow \mathbb{E}(X) = \frac{0+B}{2} = 4.$$

   (b) What is the probability that such an object lives more than 6 years on condition that it lives more than 5 years? *Solution:* $\mathbb{P}(X > 6|X > 5) = \frac{\mathbb{P}(X>6)}{\mathbb{P}(X>5)} = \frac{2/8}{3/8} = \frac{2}{3}$.

   (5 points for each part)

3. The height of a randomly chosen man follows the normal distribution with an expected value of 180 cms and a standard deviation of 10 cms. The height of a randomly chosen woman follows the normal distribution with an expected value of 170 cms and a standard deviation of 5 cms. In a large group of people 25% are men, 75% are women. You choose persons one after the other until a the person is higher than 190 cms, and then you stop choosing.

   (a) What is the probability that you make exactly 4 choices?

   Solution:

4. The hight of a randomly chosen woman is $X \sim \mathcal{N}(170, 5)$, while the hight of a randomly selected man is $Y \sim \mathcal{N}(180, 10)$. Therefore,

$$m = \mathbb{P}(Y > 190) = 1 - \Phi\left(\frac{190-180}{10}\right) = 1 - \Phi(1) \approx 0.16$$
$$w = \mathbb{P}(X > 190) = 1 - \Phi\left(\frac{190-170}{5}\right) = 1 - \Phi(4) \approx 0.00.$$

   At one choice, the probability, that the randomly selected person is taller than 190 cm’s (call it event A) is

$$p = \mathbb{P}(A) = m\cdot 0.25 + w\cdot 0.75 \approx 0.04,$$

   and the probability, that he/she is shorter than 190 cm’s is $1 - p \approx 0.96$. (Note that is also equal to $(1 - m)\cdot 0.25 + (1 - w)\cdot 0.75$.) So the probability that the above event A happens exactly at the 4th choice is

$(1-p)^3 p \approx 0.96^3 \cdot 0.04$. (Note that because we select from a large group of people, it can be considered with replacement, and use the formula for the geometric distribution.)

(b) What is the probability that the person is a woman on condition that you make exactly 4 choices?

Solution: Denote by W the event that the so (fourthly) selected person (who is the first to be taller than 190 cm's) is a woman. By the Bayes theorem,

$$\mathbb{P}(W|A) = \frac{\mathbb{P}(A, W)}{\mathbb{P}(A)} = \frac{(1-p)^3 w}{\mathbb{P}(A)} \approx 0$$

as the numerator$< w \approx 0$, but you can also feel it. (5 points for each part)

5. X is a random variable with values between $-\infty$ and 0. The density function of X is $f(x) = 3\,e^{3x}$ on the interval $-\infty; 0$.

   (a) What is the probability that $-2.5 < X < -0.5$?

   Solution: $\mathbb{P}(-2.5 < X < -0.5) = \int_{-2.5}^{-0.5} 3e^{3x} dx = e^{-1.5} - e^{-7.5}$. Alternative solution: $F(x) = e^{3x}$ if $x < 0$ and 1, otherwise. Therefore, $\mathbb{P}(-2.5 < X < -0.5) = F(-0.5) - F(-2.5) = e^{-1.5} - e^{-7.5}$.

   (b) Determine the expected value of X.

   Solution: $\mathbb{E}(X) = \int_{-\infty}^{0} x \cdot 3e^{3x} dx = -\frac{1}{3}$ with integration by parts. (5 points for each part)

6. (X, Y) follows the distribution which has the density function

$$f(x,y) = \frac{2x}{y} \quad (0 < x < 1,\ x < y < \frac{1}{x})$$

   (a) Find the density function of X.

   Solution: $f_1(x) = \int_x^{\frac{1}{x}} \frac{2x}{y} dy = 2x(\ln\frac{1}{x} - \ln x) = -4x\ln x, \quad 0 < x < 1.$

   (b) Find the conditional expected value of Y on condition that $X = x$.

   Solution: $f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)} = -\frac{1}{2y\ln x}$. So $\mathbb{E}(Y|X = x) = \int_x^{\frac{1}{x}} y f_{2|1}(y|x) dy = \frac{x - \frac{1}{x}}{2\ln x}$. (5 points for each)

7. Give the meaning of the variance of

   (a) the data set $\{1; 3; 7; 8; 11\}$ by making simple calculations (without using calculator). *(Show the details of your calculations.)*

   Solution: The sample variance of the above 5-element sample is

$$\frac{1}{5}\sum_{i=1}^{5}(X_i - \bar{X})^2 = \frac{1}{5}\sum_{i=1}^{5} X_i^2 - \bar{X}^2 = \frac{64}{5} = 12.8,$$

   where $\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ and $X_1 = 1, \ldots, X_5 = 11$.

   (b) a continuous random variable by a correct(!) mathematical formula.

   Solution: $Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, and with density: $\sigma^2 = \int (x-\mu)^2 f(x)dx = \int x^2 f(x)dx - \mu^2$, where $\mu = \int x f(x)dx$.

   In both parts we used the Steiner identity. (5 points for each)