PROBABILITY EXAM, 2016 12 19, SOLUTIONS

1. a. X can take on values 1,2,3,4. The probability mass function of X is

x	1	2	3	4
p(x)	$\frac{5}{8}$	$\frac{3}{8}\frac{5}{7}$	$\frac{3}{8}\frac{2}{7}\frac{5}{6}$	$\frac{3}{8}\frac{2}{7}\frac{1}{6}$

(Not a geometrical distribution, the sampling happens without replacement.)

b. Since 1000 is 'large', the average of the number of draws is approiximately the expectation

$$\mathbb{E}(X) = 1 \cdot \frac{5}{8} + 2 \cdot \frac{3}{8}\frac{5}{7} + 3 \cdot \frac{3}{8}\frac{2}{7}\frac{5}{6} + 4 \cdot \frac{3}{8}\frac{2}{7}\frac{1}{6} = \frac{84}{56} = 1.5$$

(5-5 points for each part)

2. a. The number (X) of the earthquakes (of any type) has Poisson distribution with parameter $2.5 + 0.5 = 3$.

$$\mathbb{P}(X = 0) = e^{-3}.$$

b. Let Y denote the number of serious earthquakes during three consecutive years. Y has Poisson distribution with parameter $3 \cdot 0.5 = 1.5$.

$$\mathbb{P}(Y > 1) = 1 - \mathbb{P}(Y = 0) - \mathbb{P}(Y = 1) = 1 - e^{-1.5} - 2.5e^{-1.5}.$$

(5-5 points for each part)

3. The hight of a randomly selected woman is $X \sim \mathcal{N}(170, 5)$, while the hight of a randomly selected man is $Y \sim \mathcal{N}(180, 10)$. Therefore,

$$p = \mathbb{P}(175 < X < 180) = \mathbb{P}(1 < \frac{X - 170}{5} < 2) = \Phi(2) - \Phi(1)$$
$$q = \mathbb{P}(175 < Y < 180) = \mathbb{P}(1 < \frac{Y - 180}{10} < 2) = \Phi(0) - \Phi(-0.5).$$

(2 points)

a. H: both have hight between 175 and 180 cm. A complete set of mutually disjoint events is: WW, WM, MW, MM (both are women,...). Then

$$\mathbb{P}(H) = \mathbb{P}(H|WW) \cdot \mathbb{P}(WW) + \mathbb{P}(H|WM) \cdot \mathbb{P}(WM) + \mathbb{P}(H|MW) \cdot \mathbb{P}(MW) + \mathbb{P}(H|MM) \cdot \mathbb{P}(MM) = p^2 \cdot 0.65^2 + 2pq0.65 \cdot 0.35 + q^2 0.35^2.$$

Observe that this is the same as $(p \cdot 0.65 + q0.35)^2$ which is an equivalently good solution. (4 points)

b. By the Bayes theorem,

$$\mathbb{P}(WW|H) = \frac{\mathbb{P}(H|WW) \cdot \mathbb{P}(WW)}{\mathbb{P}(H) \quad \text{as in part a.}}$$

(4 points)

4. a.

$$\mathbb{P}(0.5 < X < 2.5) = F(2.5) - F(0.5) = \frac{1}{27}(2.5^3 - 0.5^3)$$

(4 points)

b. We need the probability density function $f(x) = F'(x) = \frac{x^2}{9}$, $0 < x < 3$. Then

$$\mathbb{E}(X) = \int_0^3 x \cdot \frac{x^2}{9}\, dx = 2.25$$

(6 points).

5. X: weight of a sack of potatoes$\sim \mathcal{N}(10, 0.3)$.

a.

$$p = \mathbb{P}(X < 9.9) = \mathbb{P}\left(\frac{X-10}{0.3} < \frac{-0.1}{0.3}\right) = \Phi(-\frac{1}{3}) = 1 - \Phi(\frac{1}{3}).$$

b. The number of such sacks out of 5 follows binomial distribution with parameters 5 and p. Therefore, the probability that more than 2 have weight less than 9.9 is

$$\sum_{k=3}^{5} \binom{5}{k} p^k (1-p)^{5-k}.$$

(5-5 points for each part)

6. a. The density of X is

$$f_1(x) = \int_x^1 3y\, dy = \frac{3}{2}(1-x^2), \quad 0 < x < 1.$$

(4 points)

b. We need the density of Y which is

$$f_2(y) = \int_0^y 3y\, dx = 3y^2, \quad 0 < x < 1.$$

(2 points)

Then

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{3y}{3y^2} = \frac{1}{y}, \quad 0 < x < y < 1.$$

(4 points)