The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices

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Abstract. For subnormal subgroups $A \triangleleft B$ and $C \triangleleft D$ of a given group G, the factor B/A will be called *subnormally down-and-up projective* to D/C, if there are subnormal subgroups $X \triangleleft Y$ such that $AY = B$, $A \cap Y = X$, $CY = D$ and $C \cap Y = X$. Clearly, $B/A \cong D/C$ in this case. As G. Grätzer and J. B. Nation [6] have just pointed out, the standard proof of the classical Jordan-Hölder theorem yields somewhat more than widely known; namely, the factors of any two given composition series are the same up to subnormal down-and-up projectivity and a permutation. We prove the *uniqueness* of this permutation.

The main result is the analogous statement for *semimodular lattices*. Most of the paper belongs to pure lattice theory; the group theoretical part is only a simple reference to a classical theorem of H. Wielandt [14].

1. Introduction and the main results

The classical Jordan-Hölder theorem for groups goes back to C. Jordan [8] and O. Hölder [7], see also the historical remark after Theorem 5.12 in J. J. Rotman [11]. R. Dedekind [2] was certainly aware (at least for the modular case) that the Jordan-Hölder theorem followed from the corresponding lattice theoretic statement. Our goal is to strengthen this theorem, both for groups and lattices, by adding a statement on uniqueness to it. Although we formulate the Jordan-Hölder theorem in a strong but somewhat technical form, which is due to G. Grätzer and J. B. Nation [6], this form (see the first part of Theorem 1 below) can easily be extracted from the classical proofs.

As usual, the relation "*subnormal subgroup*" is the transitive closure of "normal subgroup". For subnormal subgroups $A \triangleleft B$ and $C \triangleleft D$ of a given group G, the factor B/A will be called *subnormally down-and-up projective* to D/C, if there are subnormal subgroups $X \triangleleft Y$ of G such that $AY = B$, $A \cap Y = X$, $CY = D$ and $C \cap Y = X$. Clearly, $B/A \cong D/C$ in this case, because both are isomorphic with Y/X by the Second Isomorphism Theorem.

Theorem 1. *Let* $\{1\} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_n = G$ *and* $\{1\} = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_m = G$ *be two composition series of a group* G*. Then*

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- *$n = m$, and there exists a permutation π of the set $\{1, \dots, n\}$ such that N_i/N_{i-1} is subnormally down-and-up projective to $M_{\pi(i)}/M_{\pi(i)-1}$ for all i;*
- *moreover, this permutation π is* uniquely determined*, and it has the following property: if $i, j \in \{1, \dots, n\}$ and N_i/N_{i-1} is subnormally down-and-up projective to M_j/M_{j-1}, then $j \geq \pi(i)$.*

A maximal chain of normal subgroups is called a *chief series*. Stipulating that X and Y above are normal subgroups rather than subnormal ones, we obtain the definition of *normal down-and-up projectivity*.

Theorem 2. *The same as Theorem 1 but "composition series" and "subnormally" are replaced by "chief series" and "normally" everywhere.*

Both theorems will easily follow from their lattice theoretical counterpart. Indeed, if G is a group with a finite composition series, then its subnormal subgroups form a sublattice $\mathfrak{H}(G)$ of the lattice of all subgroups by a classical result of H. Wielandt [14]; see also Theorem 1.1.5 and the remark after its proof in R. Schmidt [12], or see page 302 in M. Stern [13]. It is not hard to see that $\mathfrak{H}(G)$ is *dually* semimodular; see Theorem 2.1.8 in [12], or the proof of Theorem 8.3.3 in [13], or the proof of Theorem 9.8 in J. B. Nation [10]. Since we are going to formulate the lattice theoretical Jordan-Hölder theorem for *semimodular* lattices, as usual, "down-and-up" and "$j \geq \pi(i)$" from Theorems 1 and 2 will, of course, be dualized.

Except for a short proof at the very end, the rest of the paper is purely lattice theoretical. Basic familiarity with lattices is assumed; however, only a very small part of, say, G. Grätzer [4] or J.B Nation [10] will be needed. For intervals $[a_1, b_1]$ and $[a_2, b_2]$ of a lattice, we say that $[a_1, b_1]$ is *up-perspective* to $[a_2, b_2]$, in notation $[a_1, b_1] \nearrow [a_2, b_2]$, if $a_2 \vee b_1 = b_2$ and $a_2 \wedge b_1 = a_1$. Dually, $[a_2, b_2] \searrow [a_1, b_1]$ means $[a_1, b_1] \nearrow [a_2, b_2]$. We say that $[a_1, b_1]$ is *up-and-down projective* to $[a_2, b_2]$, in notation $[a_1, b_1] \nearrow\!\searrow [a_2, b_2]$, if there is an interval $[x, y]$ such that $[a_1, b_1] \nearrow [x, y]$ and $[x, y] \searrow [a_2, b_2]$. A lattice L is called (upper) *semimodular*, if $b \vee c$ covers or equals $a \vee c$ for all for all $a, b, c \in L$ with $a \prec b$.

Theorem 3 (Main theorem)**.** *Assume that $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ and $D = \{0 = c_0 \prec c_1 \prec \cdots \prec c_m = 1\}$ are maximal chains of a semimodular lattice L. Then*

- *$n = m$, and there is a permutation π of the set $\{1, \dots, n\}$ such that the interval $[c_{i-1}, c_i]$ is up-and-down projective to the interval $[d_{\pi(i)-1}, d_{\pi(i)}]$, for all i;*
- *moreover, this permutation π is* uniquely determined*, and it has the following property: if $i, j \in \{1, \dots, n\}$ and $[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j]$, then $j \leq \pi(i)$.*

The first part of Theorem 3 is due to G. Grätzer and J. B. Nation [6]; our contribution is the second part. In view of [6], one can say that semimodular lattices provide the foundational reason of the Jordan-Hölder theorem. Surprisingly, it will appear that the main role is played by *planar* semimodular lattices. Since these easy-to-visualize lattices and their properties are anyhow needed, we devote two lines, the proof of Corollary 16, to an entirely new approach to the first part of Theorem 3.

One may ask if $j < \pi(i)$ can happen, or if we have uniqueness for projectivities. (By *projectivity* we mean the transitive closure of the up-and-down projectivity.) The answer is given by the following example.

Remark 4. Consider M_3, the five-element modular non-distributive lattice, which is the lattice of (normal) subgroups of the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let C and D be maximal chains in M_3 such that $C \cap D = \{0, 1\}$. Then $\pi(1) = 2$ but $[c_0, c_1] \nearrow\!\searrow [d_{j-1}, d_j]$ for $j = 1, 2$. Moreover, for both permutations τ of $\{1, 2\}$, $[c_{i-1}, c_i]$ is projective to $[c_{\tau(i)-1}, c_{\tau(i)}]$, for all i.

2. Lemmas, proofs, and auxiliary lattices

A finite lattice M is called *planar*, if it has a planar diagram, that is a diagram in which the edges are non-horizontal straight lines that may intersect only at their endpoints. A planar lattice is finite by definition. Although always a fixed planar diagram is kept in mind, our statements will be valid no matter which planar diagram is considered. The edges of the (fixed) planar diagram divide the plane into regions; the minimal regions are called *cells*. The notion of cells are exemplified by the five-element non-distributive lattices: N_5 has only one cell while M_3 has two. Note that M has no cell iff it is a chain. M is said to be a *4-cell lattice*, if it is planar and each cell is surrounded by exactly four edges. Then for each cell there are $a, b \in M$, called the *left corner* and the *right corner* of the cell, such that the cell is surrounded by the edges $a \wedge b \prec a$, $a \wedge b \prec b$, $a \prec a \vee b$ and $b \prec a \vee b$, and a is on the left of b. The elements $a \wedge b$ and $a \vee b$ are called the *bottom* and the *top* of the cell, respectively. The left boundary and the right boundary of M are denoted by $\mathcal{B}_{\text{left}}(M)$ and $\mathcal{B}_{\text{right}}(M)$, respectively. Although their meaning should be clear, we mention that a rigorous technical definition is given by D. Kelly and I. Rival [9]. Note that $\mathcal{B}_{\text{left}}(M)$ and $\mathcal{B}_{\text{right}}(M)$ are maximal chains in M. By a *covering square* we mean a subset $\{a \wedge b, a, b, a \vee b\}$ such that $a \wedge b \prec a$, $a \wedge b \prec b$, $a \prec a \vee b$ and $b \prec a \vee b$. For $a \in M$, the principal ideal $[0, a] = \{x \in M : x \leq a\}$ and the principal filter $[a, 1]$ will be denoted by $\downarrow a$ and $\uparrow a$, respectively.

By a *slim lattice* we mean a finite lattice M such that $J(M)$, the poset of non-zero join-irreducible elements, contains no three-element antichain. In virtue of Dilworth [3], a finite lattice M is slim iff $J(M)$ is the union of two chains. Lemma 6 will guarantee that slim lattices are planar but we now have to assume planarity in the second part of the following lemma.

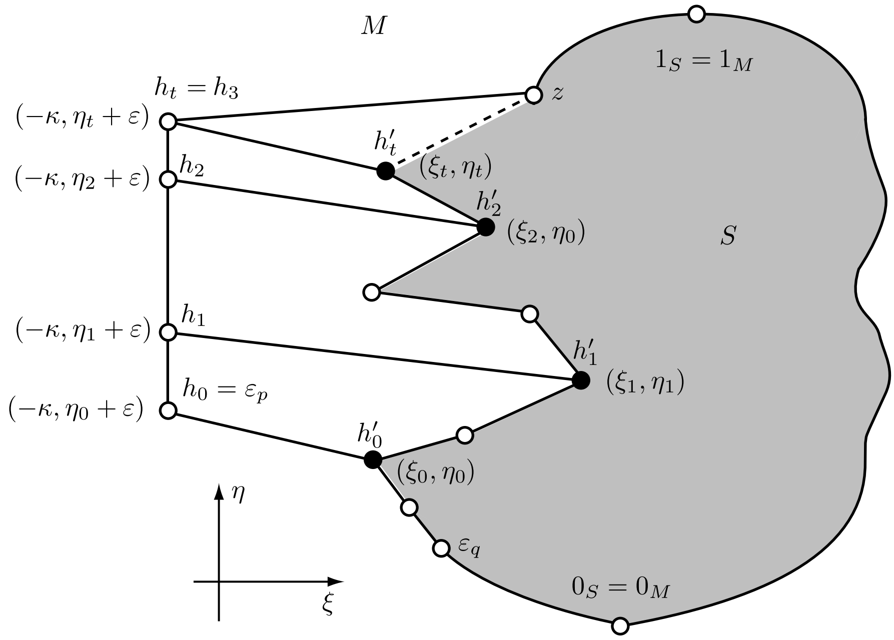

FIGURE 1. A planar extension of S to M

Lemma 5. *Let M be a slim lattice. If e is a maximal element of $J(M)$, then $\uparrow e$ is a chain. If, in addition, M is a planar lattice (with a fixed planar diagram) and e is on the left boundary of M, then $\uparrow e \subseteq \mathcal{B}_{\text{left}}(M)$.*

Proof. Assume that e is a maximal element of $J(L)$. Let $J(M) = U \cup V$ where U and V are chains. Then, say, $e \in U$. Each $x \in \uparrow e$ is of the form $x = u \vee v$ for some $u \in U$ and $v \in V$. However, then $x = e \vee x = e \vee u \vee v = e \vee v$, because e is the largest element of the chain U. So $x = e \vee v$, and any other $x' \in \uparrow e$ is $e \vee v'$ for some $v' \in V$. Since V is a chain, v and v' are comparable, whence so are x and x'. This shows that $\uparrow e$ is a chain.

Assume that, in addition, $e \in \mathcal{B}_{\text{left}}(M)$. For every $b \in \mathcal{B}_{\text{left}}(M)$ and $c \in \uparrow e$, either $e \leq b$ or $b \leq e$, since $\mathcal{B}_{\text{left}}(M)$ is a chain. The first possibility implies that b and c are comparable since $\uparrow e$ is a chain, while the second possibility implies the same trivially. Therefore $\mathcal{B}_{\text{left}}(M) \cup \{c\}$ is a chain. Since $\mathcal{B}_{\text{left}}(M)$ is a maximal chain, we get that $c \in \mathcal{B}_{\text{left}}(M)$, proving $\uparrow e \subseteq \mathcal{B}_{\text{left}}(M)$. □

Lemma 6. *Let $E = \{0 = e_0 \prec e_1 \prec \cdots \prec e_n\}$ and $F = \{0 = f_0 \prec f_1 \prec \cdots \prec f_m\}$ be non-empty chains of a finite lattice M such that $J(M) \subseteq E \cup F$. Then M has a planar diagram such that $\mathcal{B}_{\text{left}}(M) = E \cup \uparrow e_n$ and $\mathcal{B}_{\text{right}}(M) = F \cup \uparrow f_m$.*

Proof. We prove the lemma by induction on $|M|$. We assume that $|M| \geq 3$, $n \geq 1$ and $m \geq 1$, since otherwise the statement is trivial. Let $\max\bigl(J(L)\bigr)$ denote the set of maximal join-irreducible elements, note that $|\max(J(M))| \leq$

2. Since at least one of E and F contains a maximal element of $J(M)$, we can assume that $e_p \in \max\bigl(J(M)\bigr)$ for some $0 < p \le n$. Let e_q and f_r be the largest element of $\{e_1, \dots, e_{p-1}\} \cap J(M)$ and $\bigl(F \setminus \{e_p\}\bigr) \cap J(M)$, respectively, and define $E' := \{0 = e_0 \prec e_1 \prec \dots \prec e_q\}$ and $F' := \{0 = f_0 \prec f_1 \prec \dots \prec f_r\}$. Denote by S the join-subsemilattice generated by $E' \cup F'$, and let $H = M \setminus S$. Since e_p is join-irreducible, $e_p \in H$. Clearly, $M \setminus {\uparrow} e_p \subseteq S$, that is, $H \subseteq {\uparrow} e_p$. We know from Lemma 5 that ${\uparrow} e_p$ is a chain. Consequently, its elements are $\wedge$-irreducible in M. Therefore H is also a chain and its elements are $\wedge$-irreducible. This yields that S is closed with respect to meet, that is, S is a sublattice of M.

By the induction hypothesis, S has a planar diagram such that

$$\mathcal{B}_{\text{left}}(S) = E' \cup [e_q, 1_S] \text{ and } \mathcal{B}_{\text{right}}(S) = F' \cup [f_r, 1_S]. \tag{1}$$

For $x \in H$, if f_i is the largest element of ${\downarrow} x \cap F'$, then $x = e_p \vee f_i$, no matter if e_p is in F or not. So, $x' := e_q \vee f_i$ is the largest element of ${\downarrow} x \cap S$. Consider the mapping $\varphi \colon H \to S$, $x \mapsto x'$. Since $x' = \varphi(x) \in [e_q, 1_S]$, φ maps H into $\mathcal{B}_{\text{left}}(S)$. Clearly, φ is order-preserving, and it is injective since $x = e_p \vee x'$.

Observe that the chain H is a cover-preserving sublattice of M. Indeed, otherwise we would have $e_p \le x_1 < x_2 < x_3$ with $x_1, x_3 \in H$ but $x_2 \in S$. Then $x_3 = e_p \vee y_3$ for some $y_3 \in F'$ would imply $x_3 = x_2 \vee x_3 = x_2 \vee e_p \vee y_3 = x_2 \vee y_3 \in S$, a contradiction.

So, we can assume that $H = \{e_p = h_0 \prec h_1 \prec \dots \prec h_t\}$ where $t \in \mathbb{N}_0$ and the covering is understood in M. For $i = 0, \dots, t$, let $h_i' = \varphi(h_i)$ be the point (ξ_i, η_i) of the plane (in the fixed planar diagram of S). The h_i' are the black-filled elements in Figure 1, where $t = 3$. Since $h_0' < h_1' < \dots < h_t'$, we have $\eta_0 < h_1 < \dots < \eta_t$. We have to distinguish two cases.

First, we assume that $h_t \ne 1_M$. Lemma 5 and $h_t \in {\uparrow} e_p$ yield that h_t has a unique cover $z \in M$. Clearly, $z \in S$ and $1_M = 1_S$. The dotted line in Figure 1 represents $[h_t', z]$. From (1) and $h_t' \in {\uparrow} e_q$ we obtain that $[h_t', z]$ is a chain in S, and $[h_t', z]$ and all the h_i' are on the left boundary of S. In the particular case when $[h_t', z]$ is two-element, the dotted line is an edge of S that should be deleted since $h_t' < h_t < z$ in M. Clearly, if ε is a sufficiently small a positive number and κ is large enough, then positioning h_i to the point $(-\kappa, \eta_i + \varepsilon)$ keeps the planarity of the diagram, see Figure 1. This way we get a planar diagram of M.

Next, we assume that $h_t = 1_M$. Then $h_t' = 1_S$ and z is not present. However, we get a planar diagram of M in the same way as in the previous case.

Clearly, $\{e_0, \dots, e_p\} \subseteq \mathcal{B}_{\text{left}}(M)$ and $\{f_0, \dots, f_r\} \subseteq \mathcal{B}_{\text{right}}(M)$ in the planar diagram just obtained. So the statement follows from Lemma 5. ☐

Lemma 7. *For every finite lattice M, the following four conditions are equivalent:*

- *M is a slim semimodular lattice;*
- *M is a slim semimodular lattice and it is a planar 4-cell lattice;*

- *M is a planar semimodular lattice without cover-preserving M_3-sublattices;*
- *M is a planar semimodular lattice in which 4-cells and covering squares are the same.*

The third condition is clearly equivalent with the definition of G. Grätzer and E. Knapp [5]. This fact justifies a later reference to [5].

Proof. Clearly, the last two conditions are equivalent. The second condition trivially implies the first one.

Assume the first condition. Then M is planar by Lemma 6. If it contained a cover-preserving M_3-sublattice, then we could find three distinct covers v_1, v_2, v_3 of some $u \in M$ and $p_i \in \big(J(M) \cap {\downarrow} v_i\big) \setminus {\downarrow} u$ for $i \in \{1, 2, 3\}$, and $v_i = u \vee p_i$ would yield that $\{p_1, p_2, p_3\}$ is a three-element antichain in $J(L)$. Hence the third condition follows.

Assume the third condition. Then semimodularity implies that each cell of M is a 4-cell, and Corollary 2 in [1] (in particular, the first sentence of its proof) gives that $J(L)$ has no three-element antichain. That is, the second condition holds. ☐

In what follows, the notation of Theorem 3 will be fixed. In particular, L is a semimodular lattice. Let $K = (K; \vee, 0)$ be the subsemilattice of $(L; \vee, 0)$ generated by $C \cup D$. Note that $K = (K; \leq)$ is a lattice, but this auxiliary lattice is not a sublattice of L in general. However, with an appropriate choice of planar representation, we have

Lemma 8. *K, considered as a lattice, is a slim semimodular lattice with left boundary chain C and right boundary chain D. Further, $(K; \vee, 0)$ is a cover-preserving subsemilattice of $(L; \vee, 0)$.*

Proof. Clearly, $J(K) \subseteq C \cup D$. Hence K is slim. Let $x \prec y$ in K, and let $s = \max\{i : c_i \leq x\}$ and $t = \max\{j : d_j \leq x\}$. Then $x = c_s \vee d_t$, and either $y = c_{s+1} \vee d_t$ or $y = c_s \vee d_{t+1}$. In both cases, the semimodularity of L implies that $x \prec y$ in L. So, K is a cover-preserving join-subsemilattice of L, whence K is semimodular. Finally, we apply Lemma 6. ☐

The following lemma is a part of Lemma 4 of [5]. It also follows from the fact, included in the above proof, that each $x \in K$ has at most two covers.

Lemma 9 (G. Grätzer and E. Knapp [5])**.** *No two distinct 4-cells of K have the same bottom.*

By a *prime interval* we mean a two-element interval $[a, b]$, that is, an interval $[a, b]$ with $a \prec b$. The *set of all intervals* and that of *all prime intervals* of a lattice M are denoted by $\mathrm{Int}(M)$ and $\mathrm{Prin}(M)$, respectively. As usual, the *projectivity* relation on $\mathrm{Int}(M)$ is the transitive closure of "$\nearrow$" $\cup$ "$\searrow$". Projectivity should not be confused with the following notion. For $[a_0, a_1]$ and $[b_0, b_1]$ in $\mathrm{Prin}(M)$, we say that $[a_0, a_1]$ is $\mathrm{Prin}(M)$-*projective* to $[b_0, b_1]$,

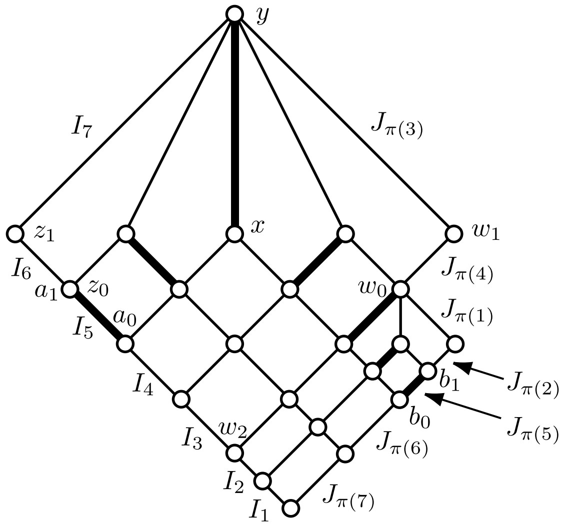

FIGURE 2. A slim semimodular lattice

if there are $k \in \mathbb{N}_0$ and $[x_i, y_i] \in \mathrm{Prin}(M)$ for $i \leq k$ such that $[a_0, a_1] = [x_0, y_0]$, $[b_0, b_1] = [x_k, y_k]$, and, for all $i \in \{1, \ldots, k\}$, $[x_{i-1}, y_{i-1}] \nearrow [x_i, y_i]$ or $[x_{i-1}, y_{i-1}] \searrow [x_i, y_i]$. One may ask if $\mathrm{Prin}(M)$-projectivity coincides with projectivity at least in particular cases. The answer is given below.

Remark 10. Let M be the principal filter $\uparrow w_2$ in Figure 2. Then M is a slim semimodular lattice. The prime intervals $[z_0, z_1]$ and $[w_0, w_1]$ are on its left boundary and right boundary, respectively. These intervals are projective in M, but they are not $\mathrm{Prin}(M)$-projective. Note also that $[z_0, z_1]$ and $[w_0, w_1]$ are projective via ten perspectivity steps but not in fewer steps.

Lemma 11. *Let M be a semimodular lattice of finite length, and let $[a_0, a_1]$, $[b_0, b_1] \in \mathrm{Prin}(M)$. Then these two prime intervals are $\mathrm{Prin}(M)$-projective iff there are $k \in \mathbb{N}_0$ and $[x_i, y_i] \in \mathrm{Prin}(M)$ for $i \leq k$ such that $[a_0, a_1] = [x_0, y_0]$, $[b_0, b_1] = [x_k, y_k]$, and, for all $i \in \{1, \ldots, k\}$, $\{x_{i-1}, y_{i-1}, x_i, y_i\}$ is a covering square.*

Proof. Assume that $[a, b], [c, d] \in \mathrm{Prin}(M)$ such that $[a, b] \nearrow [c, d]$. Take a chain $a = z_0 \prec z_1 \prec \cdots \prec z_t = b$, and define $z_i' = z_i \vee b$. Then $\{z_{i-1}, z_i, z_{i-1}', z_i'\}$ is a covering square by semimodularity. If $[a, b] \searrow [c, d]$, then $[c, d] \nearrow [a, b]$, and we obtain covering squares similarly. So, each perspectivity step gives rise to some covering squares, and the collection of all these squares prove the "only if" part. The "if" part is evident. □

Clearly, $\mathrm{Prin}(M)$-projectivity is an equivalence relation on $\mathrm{Prin}(M)$. In K, defined right before Lemma 8, the blocks (in other word, classes) of $\mathrm{Prin}(K)$-projectivity will be called *trajectories*. Let us emphasize that trajectories, unless otherwise stated, are defined and will be used only for the lattice K. The terminology is explained by the following lemma (and its proof).

Lemma 12. *The trajectories of K start at the left boundary chain C. First they go upwards (possibly in zero step), then they go downwards (possibly in zero step), and finally they reach the right boundary chain D. Trajectories never ramify.*

Proof. Take a prime interval $[x, y]$ in a trajectory T. By planarity, $[x, y]$ is the side of at most two adjacent covering squares. Indeed, it is on the left boundary of at most one square, and it is on the right boundary of at most one square. The opposite sides of these squares also belong to T. Repeating the same argument to these opposite sides and continuing to the left and to the right, we can see by Lemma 11 that T is a sequence of prime intervals such that any two consecutive prime intervals form a covering square. For example, a trajectory of the slim semimodular lattice depicted in Figure 2 is indicated by thick lines.

We can think of T as a sequence of prime intervals that "goes" from the left boundary C to the right boundary D. Since no edge is on the left boundary of two different covering squares, T cannot ramify while going to the right. However, while going from the left to the right, (segments of) T can go upwards (that is, to the northeast) or downwards (to the southwest). For example, the section from $[a_0, a_1]$ to $[x, y]$ of the trajectory in Figure 2 goes upwards, while the section from $[x, y]$ to $[b_0, b_1]$ goes downwards. Notice that a trajectory can turn down only where two distinct 4-cells have the same top.

A downward going section of T cannot be followed by an upward going section. Indeed, a down-going section could turn upwards only where two distinct 4-cells would have the same bottom, but this is impossible in virtue of Lemma 9. $\square$

Lemma 13. *For any two prime intervals of K, these intervals are up-and-down projective iff they belong to the same trajectory.*

Proof. Suppose that $[a_0, a_1], [b_0, b_1] \in \mathrm{Prin}(K)$ are up-and-down projective. Then $[a_0, a_1] \nearrow [x, y] \searrow [b_0, b_1]$ for some $[x, y] \in \mathrm{Int}(K)$. Since $x \prec y$ by semimodularity, $[a_0, a_1]$ and $[b_0, b_1]$ are $\mathrm{Prin}(K)$-projective, whence they belong to the same trajectory.

Conversely, assume that $[a_0, a_1]$ and $[b_0, b_1]$ are prime intervals belonging to the same trajectory T. By Lemma 12, the section of T between $[a_0, a_1]$ and $[b_0, b_1]$ first goes upwards (possibly in zero step), and then it goes downwards (possibly in zero step). Since the composite of up-perspectivities is an up-perspectivity, and the same holds for down-perspectivities, it follows that $[a_0, a_1] \nearrow\!\searrow [b_0, b_1]$. $\square$

Lemma 14. *If M is lattice, $[a_0, a_1], [b_0, b_1] \in \mathrm{Int}(M)$ and $[a_0, a_1] \nearrow\!\searrow [b_0, b_1]$, then $a_1 \nleq a_0 \vee b_0$ and $b_1 \nleq a_0 \vee b_0$.*

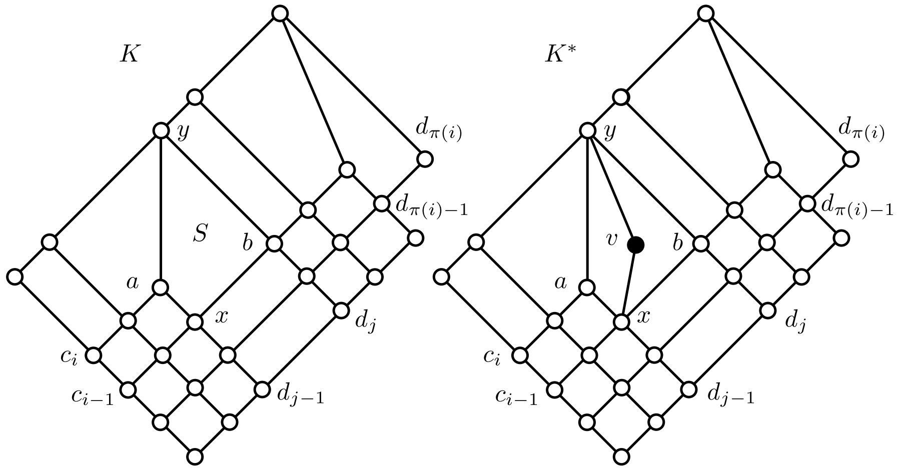

FIGURE 3. K and K^*

Proof. Choose an $[x, y] \in \mathrm{Int}(M)$ with $[a_0, a_1] \nearrow [x, y] \searrow [b_0, b_1]$. Then $a_0 \vee b_0 \le x$. If, say, $a_1 \le a_0 \vee b_0$, then $y = x \vee a_1 = x$, a contradiction. $\square$

Lemma 15. *The correspondence* $\big\{(i, \ell) : 1 \le i \le n,\ 1 \le \ell \le m$ *and* $[c_{i-1}, c_i] \nearrow\!\searrow [d_{\ell-1}, d_\ell]$ *in* $K\big\}$ *is a bijection* $\pi\colon \{1, \dots, n\} \to \{1, \dots, m\}$.

Proof. We obtain from Lemmas 12 and 13 that for each i there is exactly one ℓ with $[c_{i-1}, c_i] \nearrow\!\searrow [d_{\ell-1}, d_\ell]$. Left-right symmetry yields the converse. $\square$

Corollary 16. *The (previously known) first part of Theorem 3 holds.*

Proof. By Lemma 8, $[c_{i-1}, c_i] \nearrow\!\searrow [d_{\ell-1}, d_\ell]$ in K implies $[c_{i-1}, c_i] \nearrow\!\searrow [d_{\ell-1}, d_\ell]$ in L. Thus, Lemma 15 applies. $\square$

The bijection π defined in Lemma 15 will be called the *planar matching.* Remember that, for all $i \in \{1, \dots, n\}$,

$$[c_{i-1}, c_i] \nearrow\!\searrow [d_{\pi(i)-1}, d_{\pi(i)}] \text{ in } K \text{ and also in } L, \tag{2}$$

and these two prime intervals belong to the same trajectory of K. Note that once K is depicted, the planar matching π is very easy to find; see Figure 2, where I_i and $J_{\pi(i)}$ denote $[c_{i-1}, c_i]$ and $[d_{\pi(i)-1}, d_{\pi(i}]$, respectively.

For the rest of the paper, let us fix a pair $(i, j) \in \{1, \dots, n\}^2$ such that

$$[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j] \text{ holds in } L. \tag{3}$$

Lemma 17. *Let* $x = c_{i-1} \vee d_{j-1}$, $y = c_i \vee d_j$, $a := c_i \vee x$ *and* $b := d_j \vee x$*. If* $j \ne \pi(i)$*, then* $S = \{x, a, b, y\}$ *is a covering square in* K *with left corner* a *and right corner* b*.*

Proof. Notice that $y = a \vee b$. Since $[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j]$ in L, Lemma 14 implies that $c_i, d_j \not\le x$. Hence $x \prec a \le y$ and $x \prec b \le y$. Suppose that $a = b$. Then $[c_{i-1}, c_i] \nearrow [x, a] = [x, b] \searrow [d_{j-1}, d_j]$ and, therefore, $[c_{i-1}, c_i] \nearrow\!\searrow$

$[d_{j-1}, d_j]$ in K. Since $j \neq \pi(i)$, this contradicts (2) in virtue of Lemma 15. Hence $a \neq b$, and S is a covering square in K by semimodularity. □

Before the rigorously formulated proof of Theorem 3, we outline the idea loosely. If $[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j]$ in L and $j \neq \pi(i)$, then we insert a new element v into the 4-cell S of K defined in Lemma 17 in order to obtain a lattice K^* such that $[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j]$ already in K^*, see Figure 3. There is a unique trajectory T of K that starts at $[c_{i-1}, c_i]$ and ends at $[d_{\pi(i)-1}, d_{\pi(i)}]$. The new element ramifies T in K^*. The original part goes upwards after S within K, it may turn downwards only later, and it stops at $[d_{\pi(i)-1}, d_{\pi(i)}]$. The new part turns downwards in K^* immediately at S, then it keeps going downwards in K, and stops at $[d_{j-1}, d_j]$. This explains visually why $[d_{j-1}, d_j]$ is lower on the right boundary chain than $[d_{\pi(i)-1}, d_{\pi(i)}]$, that is, why $j \leq \pi(i)$.

Proof of Theorem 3. Corollary 16 settles the first part of Theorem 3. It suffices to show the stated "$[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j] \Rightarrow j \leq \pi(i)$" property of the planar matching π, since it clearly implies the desired uniqueness. Hence, by way of contradiction, we assume (3) together with $\pi(i) < j$.

By Lemma 7, S defined in Lemma 17 is a 4-cell in K. We add a doubly irreducible new element v to the interior of S, see Figure 3. In other words, we change S into a covering M_3 sublattice. This way we obtain a new lattice K^*.

Clearly, K^* is a 4-cell planar lattice. Trivially, or using Lemma 2 of G. Grätzer and E. Knapp [5], we get that that K^* is semimodular. Notice that K is a sublattice of K^* but, in general, $(K^*; \vee, 0)$ is not a subsemilattice of $(L; \vee, 0)$. Since $\pi(i) < j$ gives that $d_{\pi(i)} < d_j$ and we know that $d_j \leq x \vee d_j = b$, we obtain $d_{\pi(i)-1} < d_{\pi(i)} < b$. Hence $d_{\pi(i)} \leq b = b \vee d_{\pi(i)-1}$, and Lemma 14 yields that

$$[b, y] \text{ is } \underline{\textit{not}} \text{ up-and-down projective to } [d_{\pi(i)-1}, d_{\pi(i)}] \text{ in } K. \tag{4}$$

On the other hand, we know that $[c_{i-1}, c_i] \nearrow [x, a] \nearrow [b, y]$. This gives that $[c_{i-1}, c_i] \nearrow [b, y]$, implying $[c_{i-1}, c_i] \nearrow\!\searrow [b, y]$ in K. Hence Lemma 13 yields that $[b, y]$ belongs to the unique trajectory T (of K) that contains $[c_{i-1}, c_i]$. But $[d_{\pi(i)-1}, d_{\pi(i)}]$ also belongs to T by (2) and Lemma 13. Thus, $[b, y]$ and $[d_{\pi(i)-1}, d_{\pi(i)}]$ belong to the same trajectory of K, whence Lemma 13 implies $[b, y] \nearrow\!\searrow [d_{\pi(i)-1}, d_{\pi(i)}]$ in K, which contradicts (4). □

Proof of Theorems 1 and 2. Assume that G is a group with a finite composition series. As mentioned right after Theorem 2, the subnormal subgroups of G form a dually semimodular sublattice $\mathfrak{H}(G)$ of the lattice of all subgroups.

If $A, B \in \mathfrak{H}(G)$ and $A \subseteq B$, then $A \in \mathfrak{H}(B)$ by Lemma 1.1.4 of [12], see also the proof of Theorem 9.8 in [10]. Therefore the composition series of G are exactly the maximal chains of $\mathfrak{H}(G)$. Hence Theorem 1 becomes a corollary of Theorem 3. So does Theorem 2, because the lattice of normal subgroups is well-known to be modular; see, for example, Theorem 2.1.4 in R. Schmidt [12]. □

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