# CONGRUENCE-PRESERVING EXTENSIONS OF FINITE LATTICES TO ISOFORM LATTICES

G. GRÄTZER, R. W. QUACKENBUSH, AND E. T. SCHMIDT

Abstract. We call a lattice $L$ *isoform*, if for any congruence relation $\Theta$ of $L$, all congruence classes of $\Theta$ are isomorphic sublattices. In an earlier paper, we proved that for every finite distributive lattice $D$, there exists a finite isoform lattice $L$ such that the congruence lattice of $L$ is isomorphic to $D$.

In this paper, we prove a much stronger result: *Every finite lattice has a congruence-preserving extension to a finite isoform lattice.*

## 1. Introduction

Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ *uniform*, if any two congruence classes of $\Theta$ are of the same size (cardinality). Let us call the lattice $L$ *uniform*, if all congruences of $L$ are uniform. The following representation result was proved in G. Grätzer, E. T. Schmidt, and K. Thomsen [7]:

**Theorem.** *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite uniform lattice $L$.*

We introduced a much stronger concept in [6]. Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ *isoform*, if any two congruence classes of $\Theta$ are isomorphic (as lattices). Let us call the lattice $L$ *isoform*, if all congruences of $L$ are isoform.

**Theorem.** *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite isoform lattice $L$.*

Anytime we prove a representation theorem such as the two theorems quoted above, we raise the question whether a stronger form is available:

*Does every finite lattice have a congruence-preserving extension into a finite uniform/isoform lattice?*

This was raised for uniform lattices as Problem 1 in [7] and for isoform lattices as Problem 2 in [6]. (See also Problem 9 in G. Grätzer and E. T. Schmidt [5].)

In this paper, we answer these problems in the affirmative:

**Theorem 1.** *Every finite lattice $K$ has a congruence-preserving extension to a finite isoform lattice $L$.*

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This result is trivial for a finite modular lattice $K$. Indeed, $K$ is the subdirect product of simple lattices. The direct product $L$ of these lattices is an isoform, congruence-preserving extension of $K$. To generalize this to an arbitrary finite lattice $K$, we can again represent $K$ as the subdirect product of subdirectly irreducible—no longer simple—lattices. We embed each subdirectly irreducible factor into a special kind of simple lattice, and their direct product is the "cubic extension" $S$, as discussed in Section 2. Every congruence of $K$ extends to the cubic extension, but $S$ has many more congruences. The crucial step is a redefinition of the partial ordering of the cubic extension in Section 3. The poset we obtain happens to be a lattice $L$, as verified in Section 4. We compute the congruences of $L$ in Section 5, and verify that $L$ is isoform in Section 6. The proof of Theorem 1 readily follows in Section 7.

In Section 8, we follow the steps of the construction with the smallest nontrivial example.

Finally, in Section 9, we discuss various topics. First, we give two additional properties that the lattice $L$ we construct for Theorem 1 automatically has and add one more property that the lattice $L$ we construct for Theorem 1 shall have if we tweak the construction slightly. The properties are: regularity, congruence permutability, and deterministic; see Sections 9.1–9.4.

In Section 9.5, we argue the use of cubic extensions in Section 2 as opposed to the rectangular extension of G. Grätzer and E. T. Schmidt [4].

In G. Grätzer and E. T. Schmidt [6], we introduced the concept of *pruning* a poset. Our construction in this paper is based on pruning. In Section 9.6, we explain why pruning is used only implicitly in this paper.

The congruence classes in an isoform lattice are isomorphic but not in any natural fashion. In Section 9.7, we propose a definition of "naturally isoform" lattices, and we show that Theorem 1 cannot be strengthened to natural isomorphism.

Some open problems are listed in Section 9.8.

For the background of this result, we refer the reader to our survey paper [5]. For the basic concepts and notation, see [1].

## 2. Cubic extensions

Let $K$ be a finite lattice. For every meet-irreducible congruence $\Phi$ of $K$ (in formula, $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$), we form the quotient lattice $K/\Phi$, and extend it to a finite *simple* lattice $S_\Phi$ with zero $0_\Phi$ and unit $1_\Phi$. Such extensions are easy to construct; see the Lemma 7 and the discussion following it in [3].

Let $S$ be the direct product of the $S_\Phi$, $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$:

$$S = \prod (S_\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)).$$

For $a \in K$, define the vector ($\mathbf{D}$ stands for diagonal):

$$\mathbf{D}(a) = \langle a/\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K) \rangle.$$

$K$ has a natural (diagonal) embedding into $S$ by

$$\psi\colon a \mapsto \mathbf{D}(a), \quad \text{for } a \in K.$$

For a congruence $\Theta$ of $K$, let $\Theta\psi$ denote the corresponding congruence of $K\psi$. By identifying $a$ with $\mathbf{D}(a)$, for $a \in K$, we can view $S$ as an extension of $K$; we call $S$ a *cubic extension* of $K$.

The following result is a special case of Lemma 9 in [3]; we restate and reprove the result.

**Theorem 2.** *Let $K$ be a finite lattice and let $S$ be a cubic extension of $K$. Then $K$ $(= K\psi)$ has the Congruence Extension Property in $S$.*

*Note.* Recall that this means that every congruence of $K = K\psi$ extends to $S$.

*Proof.* For $\Omega \in \operatorname{Con} K$ and $\Phi \in \mathrm{M}(\operatorname{Con} K)$, define the congruence $\Gamma(\Omega, \Phi)$ on $S_\Phi$ as follows:

$$\Gamma(\Omega, \Phi) = \begin{cases} \omega, & \text{if } \Omega \leq \Phi, \\ \iota, & \text{if } \Omega \nleq \Phi; \end{cases}$$

and define

$$\mathbf{\Gamma}(\Omega) = \langle \Gamma(\Omega, \Phi) \mid \Phi \in \mathrm{M}(\operatorname{Con} K) \rangle,$$

which uniquely describes a congruence of $S$.

We show that $\mathbf{\Gamma}(\Omega)$ is an extension of $\Omega\psi$ to $S$:

$$u\psi \equiv v\psi \ (\Omega\psi) \text{ in } K\psi \quad \text{ iff } \quad u\psi \equiv v\psi \ (\mathbf{\Gamma}(\Omega)) \text{ in } S, \ \text{ for all } u, \ v \in K.$$

Assume that $u\psi \equiv v\psi$ $(\Omega\psi)$ in $K\psi$. Then $u \equiv v$ $(\Omega)$ in $K$, by the definition of $u\psi$, $v\psi$, $K\psi$, and $\Omega\psi$. The congruence $u \equiv v$ $(\Omega)$ implies that $u \equiv v$ $(\Phi)$, for all $\Phi \in \mathrm{M}(\operatorname{Con} K)$ satisfying $\Omega \leq \Phi$, that is, $u/\Phi = v/\Phi$, for all $\Phi \in \mathrm{M}(\operatorname{Con} K)$ satisfying $\Omega \leq \Phi$. This, in turn, can be written as

$$u/\Phi \equiv v/\Phi \ (\Gamma(\Omega, \Phi)), \ \text{ for all } \Phi \in \mathrm{M}(\operatorname{Con} K) \text{ satisfying } \Omega \leq \Phi,$$

since, by definition, $\Omega_\Phi = \omega$, for $\Omega \leq \Phi$.

By definition, $\Omega_\Phi = \iota$, for $\Omega \nleq \Phi$. Therefore, the congruence $u/\Phi \equiv v/\Phi$ $(\Gamma(\Omega, \Phi))$ always holds. We conclude that

$$u/\Phi \equiv v/\Phi \ (\Gamma(\Omega, \Phi)), \ \text{ for all } \Phi \in \mathrm{M}(\operatorname{Con} K).$$

The last displayed congruence is equivalent to $u\psi \equiv v\psi$ $(\mathbf{\Gamma}(\Omega))$ in $S$, which was to be proved.

Conversely, assume that $u\psi \equiv v\psi$ $(\mathbf{\Gamma}(\Omega))$ in $S$. Then

$$u/\Phi \equiv v/\Phi \ (\Gamma(\Omega, \Phi)), \ \text{ for all } \Phi \in \mathrm{M}(\operatorname{Con} K),$$

in particular,

$$u/\Phi \equiv v/\Phi \ (\Gamma(\Omega, \Phi)), \ \text{ for all } \Phi \in \mathrm{M}(\operatorname{Con} K) \text{ satisfying } \Omega \leq \Phi.$$

Thus $u/\Phi = v/\Phi$, for all $\Phi \in \mathrm{M}(\operatorname{Con} K)$ satisfying $\Omega \leq \Phi$, that is, $u \equiv v$ $(\Phi)$, for all $\Phi \in \mathrm{M}(\operatorname{Con} K)$ satisfying $\Omega \leq \Phi$. Therefore,

$$u \equiv v \quad \left(\bigwedge( \, \Phi \in \mathrm{M}(\operatorname{Con} K) \mid \Omega \leq \Phi \, )\right).$$

The lattice $\operatorname{Con} K$ is finite, so every congruence is a meet of meet-irreducible congruences, therefore,

$$\Omega = \bigwedge( \, \Phi \in \mathrm{M}(\operatorname{Con} K) \mid \Omega \leq \Phi \, ),$$

and we conclude that $u \equiv v$ $(\Omega)$ in $K$, and so $u\psi \equiv v\psi$ $(\Omega\psi)$ in $K\psi$, which was to be proved. ∎

For $\Omega \in \operatorname{Con} K$, the set

$$\Delta_\Omega = \{\, \Phi \in \mathrm{M}(\operatorname{Con} K) \mid \Omega \nleq \Phi \,\}$$

is a downset of $\mathrm{M}(\operatorname{Con} K)$, and every downset of $\mathrm{M}(\operatorname{Con} K)$ is of the form $\Delta_\Omega$, for a unique $\Omega \in \operatorname{Con} K$. The downset $\Delta_\Omega$ of $\mathrm{M}(\operatorname{Con} K)$ describes $\mathbf{\Gamma}(\Omega)$, and conversely.

We summarize the properties of the lattice $S$:

**Lemma 1.** *Let $K$ be a finite lattice with a cubic extension $S$. Then*

(i) *$S$ is finite.*

(ii) *There is a one-to-one correspondence between the subsets of $\mathrm{M}(\operatorname{Con} K)$ and the congruences $\Omega$ of $S$; the subset of $\mathrm{M}(\operatorname{Con} K)$ corresponding to the congruence $\Omega$ of $S$ is*

$$\{\, \Phi \in \mathrm{M}(\operatorname{Con} K) \mid 1_\Phi \equiv 0 \ (\Omega) \,\},$$

*where $1_\Phi$ is the unit element of the factor $S_\Phi$ of $S$. Hence, the congruence lattice of $S$ is a finite Boolean lattice.*

(iii) *Every congruence $\Omega$ of $K$ has an extension $\mathbf{\Gamma}(\Omega)$ to a congruence of $S$ corresponding to the downset $\Delta_\Omega$ of $\mathrm{M}(\operatorname{Con} K)$.*

So we see now that a cubic extension $S$ of $K$ (i) has a "cubic" congruence lattice (a power of $C_2$, an "$n$-dimensional cube"), and (ii) $K$ and its cubic extension $S$ have the same number of meet-irreducible congruences, and (iii) the congruences of $K$ extend to $S$ (but, as a rule, the cubic extension has many more congruences).

## 3. Constructing the poset $L$

Let $A$ be a finite lattice with zero 0 and unit 1. Let us call $A$ *separable*, if it has an element $v$ which is a *separator*, that is, $0 \prec v \prec 1$.

In this section, for a finite poset $P$, and a family $S_p$, for $p \in P$, of separable lattices, we construct a poset on the set $S = \prod(\, S_p \mid p \in P \,)$.

We denote by $\vee_S, \wedge_S, \leq_S, \prec_S$ the join, meet, partial ordering relation, and covering relation, respectively, in the lattice $S$, the direct product of $S_p$, for $p \in P$.

Denote by $0_p$, $1_p$, and $v_p$, the zero, the unit, and a fixed separator, respectively, of $S_p$, for $p \in P$. If there is no ambiguity, the subscripts will be dropped.

We need some notation in $S$. The elements of $S$ will be bold face lower case letters. An element $\mathbf{s} \in S$ is written in the form $\langle s_p \rangle_{p \in P}$. We write $\mathbf{s}_p$ for $s_p$. For $q \in P$, let $\mathbf{u}^q \in S$ be defined by $(\mathbf{u}^q)_q = 1$ and otherwise, $(\mathbf{u}^q)_p = 0$. For $q \in P$, let $\mathbf{v}^q \in S$ be defined by $(\mathbf{v}^q)_q = v$ and otherwise, $(\mathbf{v}^q)_p = 0$.

Let $B$ be the sublattice of $S$ generated by $\{\, \mathbf{u}^p \mid p \in P \,\}$. We call $B$ the *skeleton* of $S$; it is a boolean sublattice of $S$ with $n$ atoms. For a subset $Q$ of $P$, set $\mathbf{u}^Q = \bigvee_S \{\mathbf{u}^p \mid p \in Q\}$ with complement $(\mathbf{u}^Q)' = \mathbf{u}^{P-Q}$ in $B$ (and in $S$). The elements of $B$ are blackfilled in Figures 1 and 2.

Now we come to the crucial definition of this paper:

On the set $S$, we define a binary relation $\leq$. Let $\mathbf{a} = \langle a_p \rangle_{p \in P}, \mathbf{b} = \langle b_p \rangle_{p \in P} \in S$.

(P) $\mathbf{a} \leq \mathbf{b}$ in $L$ iff $\mathbf{a} \leq_S \mathbf{b}$ and if $p < p'$ in $P$, then

$$a_p = v_p = b_p \quad \text{implies that} \quad a_{p'} = b_{p'}.$$

Note that if $P$ is unordered, then $\leq$ is the same as $\leq_S$. In the smallest not unordered example, $P = \{p, q\}$ with $p < q$; Figure 1 illustrates what we get: $S_p = S_q = C_3$, the three-element chain; the two edges of $C_3^2$ that are not edges of $L$ are dashed.

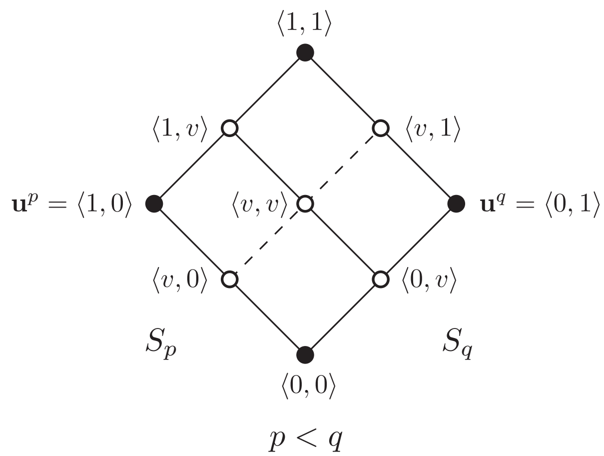


FIGURE 1. The smallest example of the construction.

In Figure 2, we show the representation of the poset $P = \{p, q, r\}$ with $p < q$ and $r < q$, with the lattices $S_p = S_q = S_r = C_3 = \{0, v, 1\}$. Four edges of $C_3^3$ are missing in $L$; on the diagram these are marked with dashed lines.

From Section 5 on, we assume that each $S_p$ is simple. So even the smallest example is fairly large. Again, let $P = \{p, q\}$ with $p < q$ and $S_p = S_q = M_3 = \{0, a, b, v, 1\}$. Figure 3 illustrates what we get; note that the diagram is turned to

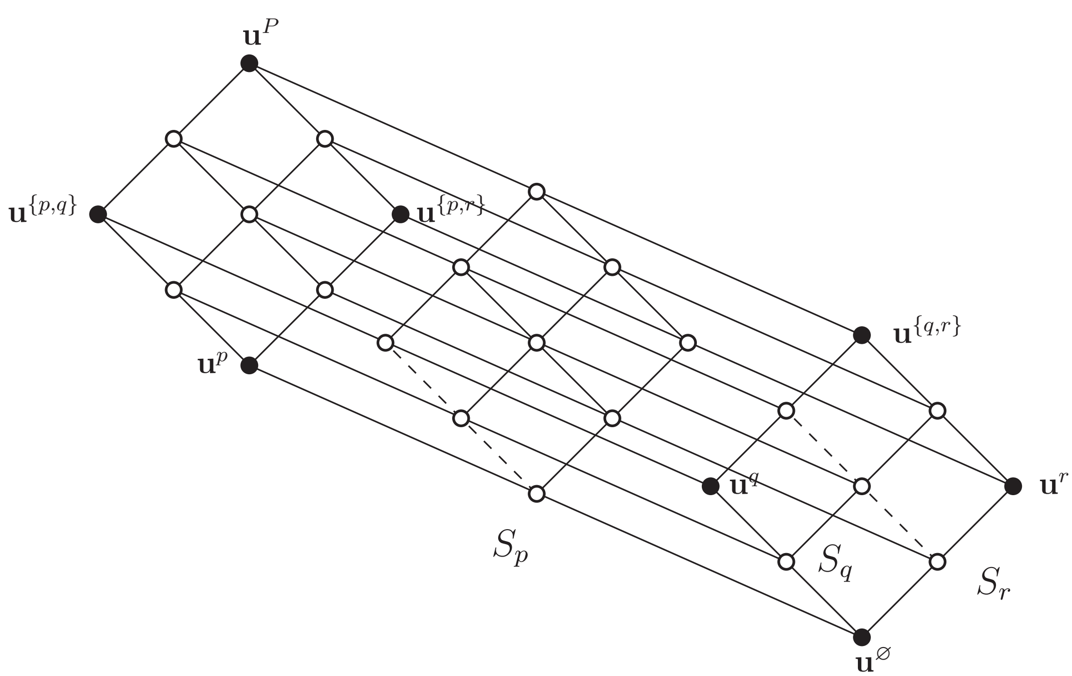


FIGURE 2. The smallest nontrivial example of the construction.

its side. In this diagram, $S_q$ is on the right, its elements are labeled 0, $a$, $b$, $v$, 1; Its unit element is $\mathbf{u}^q = \langle 0, 1\rangle$. The lattice $S_p$ is on the left, its elements are all labelled with 0; its unit element is $\mathbf{u}^p = \langle 1, 0\rangle$. Five edges of $M_3^2$ are missing in $L$; on the diagram these are marked with dashed lines.

**Theorem 3.** *Let $L$ be the relational system $\langle S, \leq\rangle$. Then $L$ is a poset.*

*Proof.* The relation $\leq$ is obviously reflexive. By the definition of $\leq$:

If $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{a} \leq_S \mathbf{b}$.

So the anti-symmetry of $\leq_S$ implies the anti-symmetry of $\leq$.

To prove the transitivity of $\leq$, let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in L$, and let $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. Then $\mathbf{a} \leq_S \mathbf{b} \leq_S \mathbf{c}$, so by the transitivity of $\leq_S$, we get $\mathbf{a} \leq_S \mathbf{c}$. To verify (P) for $\mathbf{a}$ and $\mathbf{c}$, let $p < p'$ in $P$ and assume that $a_p = v = c_p$. Since $\mathbf{a} \leq_S \mathbf{b} \leq_S \mathbf{c}$, it follows that $a_p \leq b_p \leq c_p$, and we conclude that $a_p = b_p = c_p = v$. Since $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{c}$, applying (P) twice, we obtain the equalities $a_{p'} = b_{p'}$, and $b_{p'} = c_{p'}$. It follows that $a_{p'} = c_{p'}$, verifying (P) for $\mathbf{a}$ and $\mathbf{c}$, hence proving that $\mathbf{a} \leq \mathbf{c}$. Therefore, $\leq$ is transitive. □

We shall need the following property of $L$:

**Lemma 2.**

(1) *If $\mathbf{a} \prec \mathbf{b}$ in $L$, then $\mathbf{a} \prec_S \mathbf{b}$ in $S$.*
(2) *If $\mathbf{a} \prec_S \mathbf{b}$ in $S$, then $\mathbf{a} \prec \mathbf{b}$ in $L$ unless there are $q < q'$ in $P$ with $a_q = v = b_q$ and $a_{q'} \prec b_{q'}$, in which case, $\mathbf{a} \parallel \mathbf{b}$ in $L$.*

*Proof of* (1). We start by observing that $\mathbf{a} \prec_S \mathbf{b}$ iff there is a unique $q \in P$ such that $a_q \prec b_q$ in $S_q$, and $a_r = b_r$, otherwise.

Let us assume that $\mathbf{a} \prec \mathbf{b}$ in $L$. Then $\mathbf{a} <_S \mathbf{b}$, by (P). Thus, there is $q \in P$ such that $a_q < b_q$.

*We claim* that $a_q \prec b_q$ in $S_q$. Indeed, if $a_q < c < b_q$, for some element $c \in S_q$, then we define $\mathbf{c} = \langle c_p\rangle_{p\in P}$ as follows:

$$c_p = \begin{cases} c, & \text{if } p = q; \\ v, & \text{if } a_p = v = b_p; \\ x, & \text{where } x \in \{a_p, b_p\} - \{v\}, \text{ otherwise.} \end{cases}$$

Obviously, $\mathbf{a} <_S \mathbf{c} <_S \mathbf{b}$.

We now verify that $\mathbf{a} < \mathbf{c}$. Let $p < p'$ in $P$ and let $a_p = c_p = v$. Notice that $a_p = c_p = b_p = v$. Since $\mathbf{a} < \mathbf{b}$, by (P), it follows that $a_{p'} = b_{p'}$. But then $a_{p'} = c_{p'}$, verifying $\mathbf{a} < \mathbf{c}$ by (P).

We can verify that $\mathbf{c} < \mathbf{b}$, similarly.

Therefore, $\mathbf{a} < \mathbf{c} < \mathbf{b}$, contrary to $\mathbf{a} \prec \mathbf{b}$. This proves the claim. Thus we may assume that $a_q \prec b_q$.

*We now claim* that there is only one $q$ with $a_q < b_q$. Indeed, let us assume that there are $q \neq r$ in $P$ with $a_q < b_q$ and $a_r < b_r$. We assume that $q$ is minimal in $P$ with the property: $a_q < b_q$. There are two cases to consider.

*Case 1:* $b_q = v$. Since $v$ is a separator, it follows that $a_q = 0$. Define $\mathbf{c} = \langle c_p\rangle_{p\in P}$ by

$$c_p = \begin{cases} a_p, & \text{for } p = q; \\ b_p, & \text{for } p \neq q. \end{cases}$$

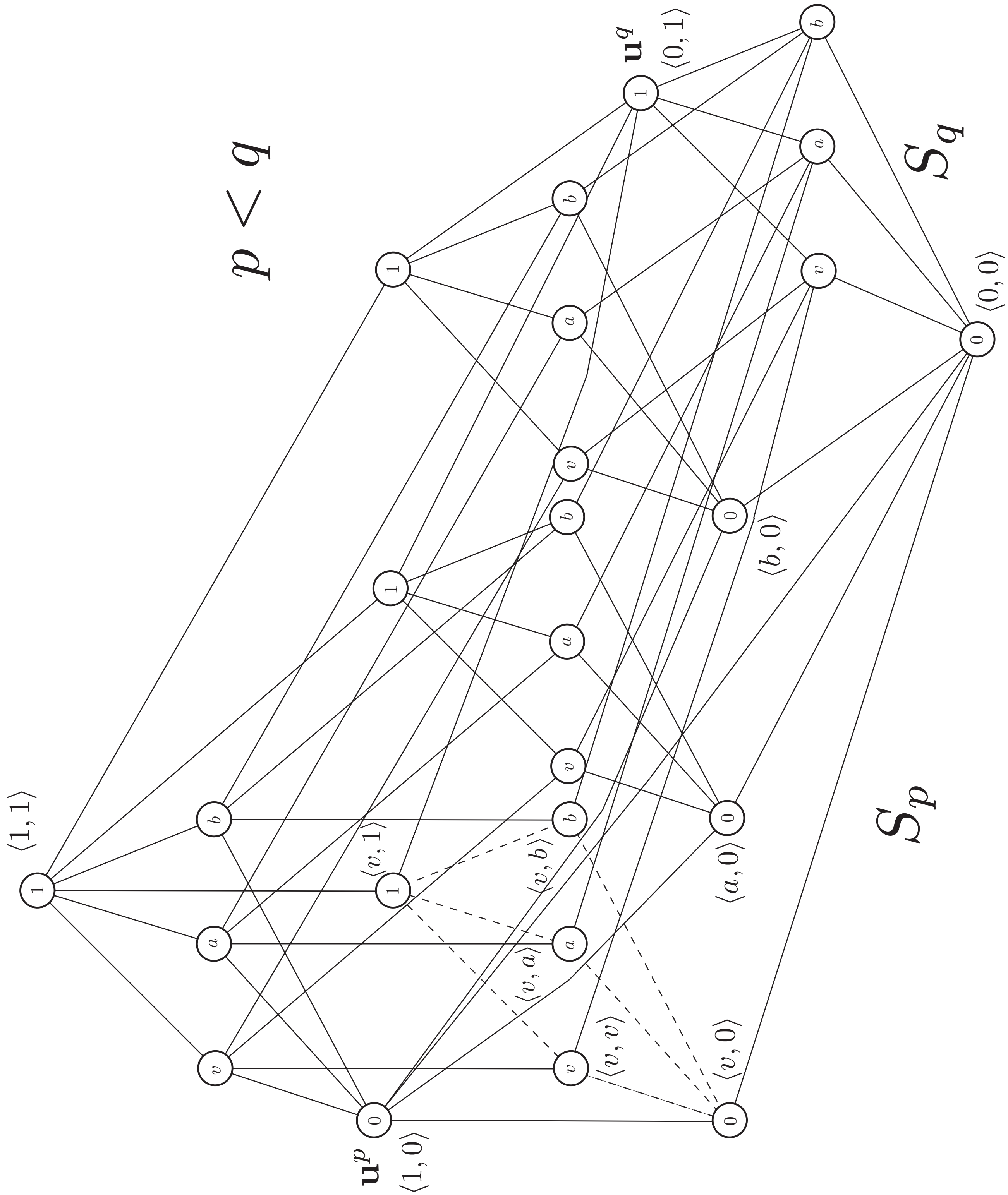


FIGURE 3. The smallest example of the construction with simple lattices—sideways.

Obviously, $\mathbf{a} <_S \mathbf{c} <_S \mathbf{b}$. Since $a_q = 0$ and $c_p = b_p$, for $p \neq q$, therefore, $c_p = v$ iff $b_p = v$. We now verify that $\mathbf{a} < \mathbf{c}$. Let $p < p'$ in $P$ and let $a_p = c_p = v$. We cannot have $p = q$, because $a_q = 0$. Thus $p \neq q$, and so $a_p = b_p = c_p = v$. Now (P) and $\mathbf{a} < \mathbf{b}$ imply that $a_{p'} = b_{p'}$. For any $p'$, either $a_{p'}$ or $b_{p'}$ equals $c_{p'}$. Therefore, $a_{p'} = b_{p'} = c_{p'}$, concluding that $\mathbf{a} < \mathbf{c}$.

We next verify that $\mathbf{c} < \mathbf{b}$. Let $p < p'$ in $P$ and let $c_p = b_p = v$ and assume to the contrary that $c_{p'} < b_{p'}$. By the definition of $\mathbf{c}$, this is possible only if $p' = q$, in which case the minimality of $q$ forces $a_p = b_p = v$, contradicting $\mathbf{a} < \mathbf{b}$. Thus $\mathbf{c} < \mathbf{b}$.

We conclude that $\mathbf{a} < \mathbf{c} < \mathbf{b}$, contradicting that $\mathbf{a} \prec \mathbf{b}$.

*Case 2:* $b_q \neq v$. In this case, define $\mathbf{c} = \langle b_p \rangle_{p \in P}$ by

$$c_p = \begin{cases} b_p, & \text{for } p = q; \\ a_p, & \text{for } p \neq q. \end{cases}$$

Obviously, $\mathbf{a} <_S \mathbf{c} <_S \mathbf{b}$. We now verify that $\mathbf{a} < \mathbf{c}$. Let $p < p'$ in $P$ and let $a_p = c_p = v$. Assume, to the contrary, that $a_{p'} \neq c_{p'}$. By the definition of $\mathbf{c}$, then $p' = q$. Since $q$ is minimal with respect to $a_q < b_q$, it follows from $p < p' = q$ that $a_p = b_p$. So $a_p = b_p = v$ and $a_{p'} < b_{p'}$, which by (P) contradicts the assumption $\mathbf{a} < \mathbf{b}$. This proves that $\mathbf{a} < \mathbf{c}$.

We next verify that $\mathbf{c} < \mathbf{b}$. Let $p < p'$ in $P$ and let $c_p = b_p = v$. By the assumption of Case 2, $b_q \neq v$, and so $p \neq q$. We conclude that $c_p = b_p = a_p = v$. Since $\mathbf{a} < \mathbf{b}$, by (P), we obtain that $a_{p'} = b_{p'}$. Therefore, $p' \neq q$. By the definition of $\mathbf{c}$, we have that $a_{p'} = c_{p'}$, so $a_{p'} = b_{p'} = c_{p'}$, verifying $\mathbf{c} < \mathbf{b}$.

We conclude that $\mathbf{a} < \mathbf{c} < \mathbf{b}$, contradicting that $\mathbf{a} \prec \mathbf{b}$.

This completes the proof of (1). □

*Proof of* (2). Now assume that $\mathbf{a} \prec_S \mathbf{b}$. Then there is a unique $q \in P$ with $a_q \prec b_q$, and otherwise, $a_r = b_r$. Consequently, if $\mathbf{a} < \mathbf{b}$, then $\mathbf{a} \prec \mathbf{b}$ in $L$. If $\mathbf{a} < \mathbf{b}$ fails, then by (P), there are elements $p < p'$ in $P$ with $a_p = v = b_p$ and $a_{p'} < b_{p'}$, implying that $\mathbf{a} \parallel \mathbf{b}$ in $L$. □

## 4. $L$ is a lattice

Now we prove that we have constructed a lattice and describe the lattice operations. To facilitate this, we introduce the following terminology: Let $\mathbf{a} = \langle a_p \rangle_{p \in P}$, $\mathbf{b} = \langle b_p \rangle_{p \in P} \in L$, and let $q \in P$; we shall call $q$ an $\{\mathbf{a}, \mathbf{b}\}$-*fork*, if $a_q = b_q = v$ and $a_{q'} \neq b_{q'}$, for some $q' > q$.

**Theorem 4.** *$L$ is a lattice. Let $\mathbf{a} = \langle a_p \rangle_{p \in P}$, $\mathbf{b} = \langle b_p \rangle_{p \in P} \in L$. Then*

$$(\mathbf{a} \vee \mathbf{b})_p = \begin{cases} 1, & \textit{if } a_p \vee b_p = v \textit{ and, for some } p' \geq p, \\ & \quad (1)\ p' \textit{ is an } \{\mathbf{a}, \mathbf{b}\}\textit{-fork, or} \\ & \quad (2)\ b_p \leq a_p \textit{ and } b_{p'} \nleq a_{p'}, \textit{ or} \\ & \quad (3)\ a_p \leq b_p \textit{ and } a_{p'} \nleq b_{p'}; \\ a_p \vee b_p, & \textit{otherwise}; \end{cases}$$

*and*

$$(\mathbf{a} \wedge \mathbf{b})_p = \begin{cases} 0, & \textit{if } a_p \wedge b_p = v \textit{ and for some } p' \geq p, \\ & \quad (1)\ \ p' \textit{ is an } \{\mathbf{a}, \mathbf{b}\}\textit{-fork, or} \\ & \quad (2)\ \ b_p \geq a_p \textit{ and } b_{p'} \ngeq a_{p'}, \textit{ or} \\ & \quad (3)\ \ a_p \geq b_p \textit{ and } a_{p'} \ngeq b_{p'}; \\ a_p \wedge b_p, & \textit{otherwise.} \end{cases}$$

*Proof.* Recall that $v$ is separable, and therefore join-irreducible, hence $a_p \vee b_p = v$ is equivalent to $\langle a_p, b_p \rangle \in \{\langle 0, v \rangle, \langle v, 0 \rangle, \langle v, v \rangle\}$ and similarly, $a_p \wedge b_p = v$ is equivalent to $\langle a_p, b_p \rangle \in \{\langle 1, v \rangle, \langle v, 1 \rangle, \langle v, v \rangle\}$. Also notice that if $a_p = b_p = v$ and Case (1) occurs, then $p$ is an $\{\mathbf{a}, \mathbf{b}\}$-fork; in Cases (2) and (3), we must have $p' > p$. Let $\mathbf{c}$ be the element defined in the join formula. To see that $\mathbf{a} \leq \mathbf{c}$ (using the definition (P) of $\leq$), first note that $\mathbf{a} \leq_S \mathbf{a} \vee_S \mathbf{b} \leq_S \mathbf{c}$, so that $\mathbf{a} \leq_S \mathbf{c}$. Let $a_p = c_p = v$ and $p' > p$; we must show that $a_{p'} = c_{p'}$.

Now $a_p = c_p = v$ implies that $a_p = a_p \vee b_p = c_p = v$, and so $b_p \leq a_p = v$. Hence, for all $q \geq p$, none of Cases (1)–(3) occur. That is, for all $q \geq p$, $q$ is not an $\{\mathbf{a}, \mathbf{b}\}$-fork (Case (1)) and $b_q \leq a_q$ (Case (2)). In Case (3), using that $b_p \leq a_p = v$, if $a_p \leq b_p$, then $a_p = b_p = v$, and since $p$ is not an $\{\mathbf{a}, \mathbf{b}\}$-fork, $a_q = b_q$, for $q \geq p$.

Consequently, $b_{p'} \leq a_{p'}$. Hence, $a_{p'} = a_{p'} \vee b_{p'} \leq c_{p'}$, with equality holding unless $v = a_{p'} = a_{p'} \vee b_{p'}$ and one of Cases (1)–(3) holds, for some $p'' \geq p'$. But Case (1) cannot occur because $p'' = q \geq p$ is not an $\{\mathbf{a}, \mathbf{b}\}$-fork, and Case (2) cannot occur since $p'' \geq p$ forces $b_{p''} \leq a_{p''}$. For Case (3), note that $b_{p'} \leq a_{p'}$, since $p' > p$. Hence, $a_{p'} \leq b_{p'} = v$ forces $a_{p'} = b_{p'} = v$. But $p' > p$ is not an $\{\mathbf{a}, \mathbf{b}\}$-fork, so $a_{p''} = b_{p''}$, and so Case (3) cannot occur. Thus, $a_{p'} = c_{p'}$ and $\mathbf{a} \leq \mathbf{c}$. Similarly, $\mathbf{b} \leq \mathbf{c}$.

It remains to show that $\mathbf{c}$ is the least upper bound of $\mathbf{a}$, $\mathbf{b}$ in $L$. So let $\mathbf{a}$, $\mathbf{b} \leq \mathbf{d}$. Then $\mathbf{a}$, $\mathbf{b} \leq_S \mathbf{d}$, so $\mathbf{a} \vee_S \mathbf{b} \leq_S \mathbf{d}$. We first show that $\mathbf{c} \leq_S \mathbf{d}$. For $p \in P$, if $a_p \vee b_p = c_p$, then $c_p \leq d_p$. Let $a_p \vee b_p < c_p$; as in the previous paragraph, it follows that $a_p \vee b_p = v < 1 = c_p$. Therefore, $v \leq d_p \leq 1$. If $d_p = 1$, then $c_p \leq d_p$, as required. Now assume that $d_p = v$. Since $a_p \vee b_p = v < 1 = c_p$, one of Cases (1)–(3) must occur; in each case, we will arrive at a contradiction. If Case (1) occurs, then $a_p = b_p = d_p = v$ and, for some $p' > p$, we have $a_{p'} \neq b_{p'}$. We conclude that $d_{p'}$ cannot equal both $a_{p'}$ and $b_{p'}$, contradicting that $\mathbf{d}$ is a common upper bound of $\mathbf{a}$ and $\mathbf{b}$. If Case (2) occurs, then $b_p \leq a_p = v = d_p$ and $b_{p'} \nleq a_{p'}$, for some $p' > p$. Then $d_{p'} \geq a_{p'} \vee b_{p'} > a_{p'}$, contradicting that $\mathbf{a} \leq \mathbf{d}$. Symmetrically, Case(3) leads to a contradiction. Thus, $\mathbf{c} \leq_S \mathbf{d}$.

To prove that $\mathbf{c} \leq \mathbf{d}$, let $c_p = d_p = v$ and choose $p < p'$; we must show that $c_{p'} = d_{p'}$. Since $c_p = d_p = v$, the join formula tells us that $a_p \vee b_p = c_p = v$; without loss of generality, we may assume that $a_p = v$ so that $a_p = v = d_p$. As $\mathbf{c} \leq_S \mathbf{d}$, either $c_{p'} = d_{p'}$ or $c_{p'} < d_{p'}$. In the latter case, we have $a_{p'} \leq c_{p'} < d_{p'}$, contradicting that $\mathbf{a} \leq \mathbf{d}$. Hence, $c_{p'} = d_{p'}$. This proves that $\mathbf{c} \leq \mathbf{d}$ and therefore $\mathbf{c}$ is the least upper bound of $\mathbf{a}$, $\mathbf{b}$ in $L$.

The proof of the meet formula is similar, *mutatis mutandis.* ∎

From now on, for $\mathbf{a}$, $\mathbf{b} \in L$, $\mathbf{a} \leq \mathbf{b}$, $\mathbf{a} \vee \mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$, refer to the partial ordering and operations in $L$.

We shall need two lemmas that immediately follow from the join and meet formulas.

For $p \in P$, the element $v$ is doubly irreducible in $S_p$. Therefore, each $S'_p = S_p - \{v\}$ is a sublattice of $S_p$. Let $S'$ be the direct product of the $S'_p$, $p \in P$. Obviously, $S'$ is a sublattice of $S$.

**Lemma 3.**

(1) *$S'$ is a sublattice of both $L$ and $S$.*
(2) *For $p \in P$, the interval $[0, \mathbf{u}^p]$ is a sublattice of both $L$ and $S$; in fact, the same sublattice, that is, $(\leq)|_{[0,\mathbf{u}^p]} = (\leq_S)|_{[0,\mathbf{u}^p]}$.*

*Proof.* The proof follows from the join and meet formulas of Theorem 4. □

**Lemma 4.** *Let $\mathbf{a} \in L$ and let $D$ be a downset of $P$. Then*

$$\mathbf{a} \vee \mathbf{u}^D = \mathbf{a} \vee_S \mathbf{u}^D,$$

*and*

$$\mathbf{a} \wedge (\mathbf{u}^D)' = \mathbf{a} \wedge_S (\mathbf{u}^D)'.$$

*Proof.* By the join formula, $(\mathbf{a} \vee \mathbf{u}^D)_p = (\mathbf{a} \vee_S \mathbf{u}^D)_p$ unless $a_p = v$, $p \notin D$, and $a_{p'} < 1$, for some $p < p'$ with $p' \in D$. But this cannot happen since $D$ is a downset. By the meet formula, $(\mathbf{a} \wedge (\mathbf{u}^D)')_p = (\mathbf{a} \wedge_S (\mathbf{u}^D)')_p$ unless $a_p = v$, $p \notin D$, and $a_{p'} > 0$, for some $p < p'$ with $p' \in D$. But this also cannot happen since $D$ is a downset. □

## 5. The congruences of $L$

*Now let each $S_p$ be simple;* then $S$ has a congruence lattice isomorphic to $B$ such that $\mathbf{u}^Q \in B$ is associated with the smallest congruence on $S$ collapsing 0 and $\mathbf{u}^Q$, denoted $\Theta_Q$. Clearly, $S$ is isoform. We are going to prove that the congruences of $L$ are just the congruences $\Theta_D$ of $S$ for $D$ a downset of $P$. As an illustration, in Figure 3, the lattice has exactly one nontrivial congruence, "projecting" $L$ onto $M_q$. The five congruence classes have five elements each, labelled $x$, with $x \in \{0, v, a, b, 1\}$.

For $p \in P$, the lattice $S_p$ is assumed to be simple, so the set $S_p - \{0, 1, v\} \neq \varnothing$. Therefore, we can select $w_p \in S_p - \{0, 1, v\}$; we shall write $w$ for $w_p$ if the index is understood. We define $\mathbf{w}^q$ by $(\mathbf{w}^q)_q = w$ and otherwise, $(\mathbf{w}^q)_p = 0$.

**Lemma 5.** *Let $\Theta$ be a congruence of $L$, and let $0 \equiv \mathbf{u}^p \ (\Theta)$, for some $p \in P$. If $q < p$, then $0 \equiv \mathbf{u}^q \ (\Theta)$.*

*Proof.* From $0 \equiv \mathbf{u}^p \ (\Theta)$ and $q < p$, the join formula tells us that $\mathbf{v}^q = 0 \vee \mathbf{v}^q \equiv \mathbf{u}^p \vee \mathbf{v}^q = \mathbf{u}^{\{p,q\}} \ (\Theta)$. Similarly, $0 = \mathbf{w}^q \wedge \mathbf{v}^q \equiv \mathbf{w}^q \wedge \mathbf{u}^{\{p,q\}} = \mathbf{w}^q \ (\Theta)$. Lemma 3 implies that $[0, \mathbf{u}^q]$ is a simple sublattice of $L$, so we obtain $0 \equiv \mathbf{u}^q \ (\Theta)$. □

**Lemma 6.** *Let $\Theta$ be a congruence of $L$ and $D$ a downset of $P$. Let $\mathbf{a}$, $\mathbf{b} \in L$ with $a_p = 0$ and $b_p = 1$, for $p \in D$ and $a_p = b_p$, for $p \notin D$. Then $\mathbf{a} \leq \mathbf{b}$ and $0 \equiv \mathbf{u}^D$ $(\Theta)$ iff $\mathbf{a} \equiv \mathbf{b} \ (\Theta)$.*

*Proof.* Clearly, $\mathbf{b} = \mathbf{a} \vee_S \mathbf{u}^D = \mathbf{a} \vee \mathbf{u}^D$, by Lemma 3. Thus, $\mathbf{a} \leq \mathbf{b}$. Let $0 \equiv \mathbf{u}^D \ (\Theta)$. Then $\mathbf{a} = 0 \vee \mathbf{a} \equiv \mathbf{u}^D \vee \mathbf{a} = \mathbf{b} \ (\Theta)$. Conversely, if $\mathbf{a} \equiv \mathbf{b} \ (\Theta)$, then $0 = \mathbf{a} \wedge \mathbf{u}^D \equiv \mathbf{b} \wedge \mathbf{u}^D = \mathbf{u}^D \ (\Theta)$. □

**Lemma 7.** *Let $\Theta$ be a congruence of $L$ and let $\mathbf{a} \prec \mathbf{b}$ in $L$ with $\mathbf{a} \equiv \mathbf{b} \ (\Theta)$. Then there is a unique $p \in P$ with $a_p \prec b_p$ and $a_q = b_q$, for $p \neq q$; moreover, $0 \equiv \mathbf{u}^p \ (\Theta)$.*

*Proof.* If $\mathbf{a} \prec \mathbf{b}$, then from Lemma 2, there is a unique $p \in P$ with $a_p \prec b_p$ and $a_q = b_q$ otherwise. Moreover, if $a_q = b_q = v$, then $q \nleq p$. If $b_p = v$ so that $a_p = 0$, then we get $(\mathbf{a} \vee \mathbf{w}^p)_p = w < 1 = (\mathbf{b} \vee \mathbf{w}^p)_p$ and otherwise $(\mathbf{a} \vee \mathbf{w}^p)_q = a_q = b_q = (\mathbf{b} \vee \mathbf{w}^p)_q$ since $a_q = b_q = v$ implies that $q \nleq p$, so that $(\mathbf{w}^p)_r = 0$, for all $q < r$. This yields that $\mathbf{a} \vee \mathbf{w}^p \prec \mathbf{b} \vee \mathbf{w}^p$ with $\mathbf{a} \vee \mathbf{w}^p \equiv \mathbf{b} \vee \mathbf{w}^p \ (\Theta)$. Thus, we may assume that $b_p \neq v$. Then $0 \leq \mathbf{a} \wedge \mathbf{u}^p < \mathbf{b} \wedge \mathbf{u}^p \leq \mathbf{u}^p$ and $\mathbf{a} \wedge \mathbf{u}^p \equiv \mathbf{b} \wedge \mathbf{u}^p \ (\Theta)$. By Lemma 3, $[0, \mathbf{u}^p]$ is a simple sublattice of $L$. Hence, $0 \equiv \mathbf{u}^p \ (\Theta)$. ☐

**Theorem 5.** *Let $\Theta$ be a congruence of $L$; then there is a downset $D$ of $P$ such that $\Theta = \Theta_D$. Conversely, let $D$ be a downset of $P$; then $\Theta_D$ is a congruence of $L$.*

*Proof.* Let $\Theta$ be a congruence of $L$. Define $\mathrm{Prec}(\Theta)$ to be the set of all $\langle \mathbf{a}, \mathbf{b} \rangle \in L^2$ such that $\mathbf{a} \equiv \mathbf{b} \ (\Theta)$ and $\mathbf{a} \prec \mathbf{b}$. Define $D$ to be the set of all $p \in P$ such that $a_p \prec b_p$ for some $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathrm{Prec}(\Theta)$. By Lemma 7, $0 \equiv \mathbf{u}^p \ (\Theta)$, for all $p \in D$. By Lemma 5, $D$ is a downset of $P$. By Lemma 3, $\bigvee(\, \mathbf{u}^p \mid p \in D\,) = \mathbf{u}^D$; hence, $0 \equiv \mathbf{u}^D$ $(\Theta)$. By Lemma 6, $\Theta_D \subseteq \Theta$. On the other hand, since $L$ is finite, $\Theta$ is the smallest equivalence relation containing $\mathrm{Prec}(\Theta)$, so we must have $\Theta \subseteq \Theta_D$. Thus, $\Theta = \Theta_D$.

Conversely, let $D$ be a downset of $P$. A typical $\Theta_D$-class is of the form $[\mathbf{a}, \mathbf{b}]$, where for $p \in D$, $a_p = 0$ and $b_p = 1$, and otherwise $a_q = b_q$. By Lemma 4, $\mathbf{a} \leq \mathbf{a} \vee \mathbf{u}^D = \mathbf{a} \vee_S \mathbf{u}^D = \mathbf{b}$. Let $\mathbf{c} \in L$; it suffices to show that $\mathbf{a} \vee \mathbf{c} \equiv \mathbf{b} \vee \mathbf{c} \ (\Theta)$ and $\mathbf{a} \wedge \mathbf{c} \equiv \mathbf{b} \wedge \mathbf{c} \ (\Theta)$.

For the join, we may take $\mathbf{a} \leq \mathbf{c}$, so that $\mathbf{a} \vee \mathbf{c} = \mathbf{c} \leq \mathbf{b} \vee \mathbf{c}$. Let us assume that $c_p < (\mathbf{b} \vee \mathbf{c})_p$; we must show that $p \in D$. If $c_p < b_p \vee c_p$, then from $a_p \leq c_p$, we conclude that $a_p \neq b_p$, and so $p \in D$. Otherwise, $c_p = b_p \vee c_p < (\mathbf{b} \vee \mathbf{c})_p$ so that $c_p = v = b_p \vee c_p < 1 = (\mathbf{b} \vee \mathbf{c})_p$. Again if $a_p \neq b_p$, then $p \in D$; so assume that $a_p = b_p \leq c_p = v$. As $D$ is a downset, we must have $a_q = b_q$ for all $q \geq p$. Now, $b_p \vee c_p = v < 1 = (\mathbf{b} \vee \mathbf{c})_p$ implies that one of Cases (1)–(3) holds for $\{\mathbf{b}, \mathbf{c}\}$, for some $p' \geq p$. Note that $b_p \leq c_p$ implies that Case(3) cannot occur. If Case (1) occurs, then $a_{p'} = b_{p'} = c_{p'} = v$; hence, $p'$ being a $\{\mathbf{b}, \mathbf{c}\}$-fork forces $p'$ to be an $\{\mathbf{a}, \mathbf{c}\}$-fork, contradicting that $\mathbf{a} \leq \mathbf{c}$. If Case (2) occurs, then $a_{p'} = b_{p'} \nleq c_{p'}$, again contradicting $\mathbf{a} \leq \mathbf{c}$. This proves that $\mathbf{a} \vee \mathbf{c} \equiv \mathbf{b} \vee \mathbf{c} \ (\Theta)$.

For the meet, we proceed similarly. ☐

## 6. $L$ is isoform

As in Section 5, we assume that *each $S_p$ is simple.*

**Theorem 6.** *$L$ is isoform.*

*Proof.* Let $D$ be a downset of $P$; in $S$ every block of $\Theta_D$ can be written in the form $[\mathbf{a}, \mathbf{a} \vee_S \mathbf{u}^D]$, for some $\mathbf{a} \leq_S (\mathbf{u}^D)'$. Hence if $\mathbf{b}, \mathbf{c} \in [\mathbf{a}, \mathbf{a} \vee_S \mathbf{u}^D]$ and $b_p \neq c_p$, then $p \in D$. The map $\varphi_{\mathbf{a}} \colon [0, \mathbf{u}^D] \to [\mathbf{a}, \mathbf{a} \vee_S \mathbf{u}^D]$ defined by $\phi_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} \vee_S \mathbf{u}^D$ is an isomorphism of sublattices of $S$. By Lemma 4, $\mathbf{a} \leq \mathbf{a} \vee \mathbf{u}^D = \mathbf{a} \vee_S \mathbf{u}^D$; thus, the blocks of $\Theta_D$ in $L$ are also of the form $[\mathbf{a}, \mathbf{a} \vee_S \mathbf{u}^D]$ for some $\mathbf{a} \leq (\mathbf{u}^D)'$ (which is the same as $\mathbf{a} \leq_S (\mathbf{u}^D)'$). We shall show that $\phi_{\mathbf{a}}$ is also an isomorphism of $L$-sublattices; this will prove that $L$ is isoform. We shall make use without further mention that for $\mathbf{b}, \mathbf{c} \in L$, $(\mathbf{b} \vee_S \mathbf{c})_p = b_p \vee c_p$, for all $p \in P$.

Obviously, $\phi_{\mathbf{a}}$ is a bijection. Let us assume that $0 \leq \mathbf{b} \prec \mathbf{c} \leq \mathbf{u}^D$. Then by Lemma 4 and the fact that $\phi_{\mathbf{a}}$ is an $S$-isomorphism, it follows that $\mathbf{b} \vee_S \mathbf{a} \prec_S \mathbf{c} \vee_S \mathbf{a}$. If we do not have $\mathbf{b} \vee_S \mathbf{a} \prec \mathbf{c} \vee_S \mathbf{a}$, then by Lemma 2, there are $p < p'$ with $b_p \vee a_p = v = c_p \vee a_p$ and $b_{p'} \vee a_{p'} \prec c_{p'} \vee a_{p'}$. From this latter, we get $p' \in D$;

since $D$ is a downset and $p < p'$, we have $p \in D$. But then $a \leq (u^D)'$ implies that $a_p = a_{p'} = 0$; therefore, $v = b_p \vee a_p = b_p$ and $v = c_p \vee a_p = c_p$, so that $b_p = c_p = v$. Since $\mathbf{b} \prec \mathbf{c}$, we have that $b_{p''} = c_{p''}$ for all $p < p''$. For $p'' = p'$, this yields $b_{p'} \vee a_{p'} = b_{p'} = c_{p'} = c_{p'} \vee a_{p'}$, contradicting that $b_{p'} \vee a_{p'} \prec c_p \vee a_p$. Thus, $\mathbf{b} \vee_S \mathbf{a} \prec \mathbf{c} \vee_S \mathbf{a}$.

Now assume that $\mathbf{a} \leq \mathbf{b}' \prec \mathbf{c}' \leq \mathbf{a} \vee_S \mathbf{u}^D$; hence, $\mathbf{a} \leq_S \mathbf{b}' \prec_S \mathbf{c}' \leq_S \mathbf{a} \vee_S \mathbf{u}^D$. Since $\varphi_a$ is an $S$-isomorphism, there are unique $0 \leq_S \mathbf{b} \prec_S \mathbf{c} \leq_S \mathbf{u}^D$ such that $\mathbf{b}' = \mathbf{b} \vee_S \mathbf{a}$ and $\mathbf{c}' = \mathbf{c} \vee_S \mathbf{a}$. Hence there is a unique $p \in D$ such that $b_p \prec c_p$, $b_p \vee a_p \prec c_p \vee a_p$, and $b_q \vee a_q = c_q \vee a_q$, for $p \neq q$. We need to show that $\mathbf{b} \leq \mathbf{c}$. So let $b_r = v = c_r$; then $r \neq p$. We have to verify that $b_q = c_q$, for $r < q$. This can fail only if $r < p$. But then $r < p \in D$, a downset; hence, $r \in D$. Since $\mathbf{a} \leq (\mathbf{u}^D)'$, it follows that $a_r = a_p = 0$. But then $b_r \vee a_r = b_r = v = c_r = c_r \vee a_r$, while $b_p \vee a_p \prec c_p \vee a_p$, contradicting that $\mathbf{b}' \leq \mathbf{c}'$. Thus $\mathbf{b} \prec \mathbf{c}$. This completes the proof that $L$ is isoform. $\square$

## 7. Proof of Theorem 1

Let $K$ be a finite lattice. As in Section 2, for every meet-irreducible congruence $\Phi$ of $K$, we form the quotient lattice $K/\Phi$, and extend it to a finite, *simple*, separable lattice $S_\Phi$. The cubic extension $S$ is the direct product of the $S_\Phi$, $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$ and $\psi \colon a \mapsto \mathbf{D}(a) = \langle a/\Phi \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K) \rangle$ is the diagonal embedding. For a congruence $\Theta$ of $K$, let $\Theta\psi$ denote the corresponding congruence of $K\psi$. We identify $a$ with $\mathbf{D}(a)$, for $a \in K$.

Note that by Lemma 3, the set $\{\, \mathbf{D}(a) \mid a \in K \,\}$ is also a sublattice of $L$, so we can regard $\psi$ an embedding of $K$ into $L$.

For $\Theta \in \mathrm{Con}\, K$ and $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$, let $\Gamma(\Theta, \Phi) = \omega$, if $\Theta \leq \Phi$, and $\Gamma(\Theta, \Phi) = \iota$, otherwise. Then $\mathbf{\Gamma}(\Theta) = \langle \Gamma(\Theta, \Phi) \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K) \rangle$ is a congruence of $S$, an extension of $\Theta\psi$.

Now carry out the construction of $L$ as in Section 3 with the poset $P = \mathrm{M}(\mathrm{Con}\, K)$ and for the lattices $S_\Phi$, $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$.

In Lemma 1.(iii), we observe that the congruences $\mathbf{\Gamma}(\Theta)$ of $S$ correspond to the downsets of $\mathrm{M}(\mathrm{Con}\, K)$.

In Section 5, we describe the congruences of $L$ as $\Theta_D$, where $D$ be a downset of $P$ (see Theorem 5).

Now comes the trivial but crucial observation:

We assigned in Section 2 to a congruence $\Omega$ of $K$, the downset

$$D_\Omega = \{\, \Phi \in \mathrm{M}(\mathrm{Con}\, K) \mid \Phi \nleq \Omega \,\}.$$

Then the binary relation $\mathbf{\Gamma}(\Omega)$ is the same as the binary relation $\Theta_{D_\Omega}$.

Now to prove Theorem 1, identify $a$ with $\mathbf{D}(a)$, for $a \in K$. Let $\Theta$ be a congruence of $K$. Then $\mathbf{\Gamma}(\Theta)$ by the above observation is a congruence of $L$, so every congruence of $K$ extends to a congruence $\mathbf{\Gamma}(\Theta)$ of $L$. But by Theorem 5 and the above observation, every congruence of $L$ is of the form $\mathbf{\Gamma}(\Theta)$, hence $\Theta \mapsto \mathbf{\Gamma}(\Theta)$ is an isomorphism between $\mathrm{Con}\, K$ and $\mathrm{Con}\, L$. Therefore, $\mathbf{\Gamma}(\Theta)$ is the unique extension of $\Theta$ to $L$, so $L$ is a congruence-preserving extension. This completes the proof of Theorem 1.

## 8. A small example

What is the lattice $L$ we obtain in Theorem 1, if we start with a small lattice $K$? In this section, we construct $L$ for the lattice $K$ of Figure 4. The lattice $K$ has only one nontrivial congruence $\Theta$, splitting $K$ into two classes, indicated by the dashed line. It has two meet-irreducible congruences, $\Theta$ and $\omega$.

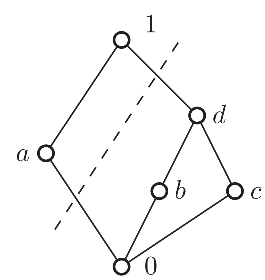


FIGURE 4. The lattice $K$.

To construct the cubic extension of $K$, we form $K/\Theta$ and $K = K/\omega$ and we have to embed them into finite, simple, separable lattices, $S_\Theta$ and $S_\omega$, respectively. The choice for $S_\omega$ is clear, see Figure 5. We just add one element $v$ to $K$ to make it simple.

However, the choice for $S_\Theta \cong C_2$ is problematic. The smallest simple separable lattice containing the two-element chain is $M_3$. Choosing it, would make $S$ a difficult to draw 35 element lattice. So we use a trick: We declare that we can choose the two-element chain as $S_\Theta$. This, of course, does not have a separating element. However, the definition of the partial ordering can be rewritten:

(P) $\mathbf{a} \leq \mathbf{b}$ in $L$ iff $\mathbf{a} \leq_S \mathbf{b}$ and if $p < p'$ in $P$, then

$$a_p = v_p = b_p \quad \text{implies that} \quad a_{p'} = b_{p'},$$

provided that $v_p$ exists.

We do not run into trouble with this until Section 5, in particular, Lemma 5, where if $p < q$ in $P$, then we need the existence of $v_p$.

So in our example, we need a separating element in $S_\omega$ but not in $S_\Theta$, so we can choose $S_\Theta = C_2$, the two-element chain. This reduces the number of elements in $S$ to 14, so we can easily draw that; see Figure 6. We also indicate how $K$ is a sublattice of $S$.

Finally, we construct $L$ from $S$ by deleting a single edge, see Figure 7. We also indicate how $K$ is a sublattice of $L$, so that it can be readily verified that $L$ is a congruence-preserving extension of $K$.

## 9. Discussion

9.1. **Regular lattices.** Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ *regular*, if any congruence class of $\Theta$ determines the congruence. Let us call the lattice $L$ *regular*, if all congruences of $L$ are regular.

Sectionally complemented lattices are regular, so we already have a representation theorem in G. Grätzer and E. T. Schmidt [2]; we also have the stronger congruence-preserving extension version in G. Grätzer and E. T. Schmidt [4]. We

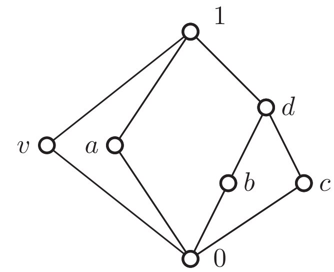


FIGURE 5. The lattice $S_\omega$.

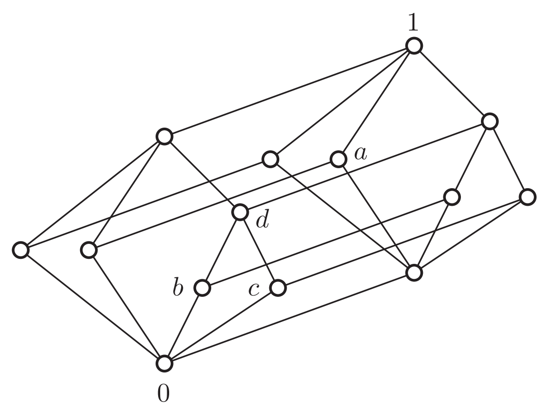


FIGURE 6. The lattice $S$.

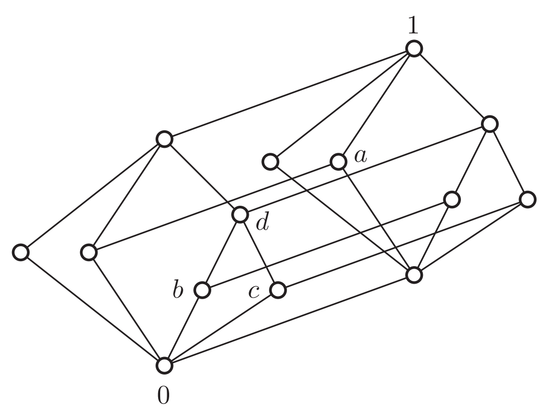


FIGURE 7. The lattice $L$.

would like to point out, that Theorem 1 contains these statements. This follows from

**Lemma 8.** *Every finite isoform lattice is regular.*

*Proof.* Let $L$ be an isoform lattice. Let $\Theta$ and $\Phi$ congruences sharing the block $A$. Then $A$ is also a block of $\Theta \wedge \Phi$; in other words, we can assume that $\Theta \leq \Phi$. Let $B$ be an arbitrary $\Theta$ block. Since $\Theta \leq \Phi$, there is a (unique) $\Phi$ block $\overline{B}$ containing $B$. Since $L$ is isoform, $A \cong B$ and $A \cong \overline{B}$; by finiteness, $B = \overline{B}$, that is, $\Theta = \Phi$. □

**Corollary.** *The lattice $L$ of Theorem 1 is regular.*

9.2. **Permutable congruences.** The representation result for permutable congruences appeared in G. Grätzer and E. T. Schmidt [2]. Since sectionally complemented lattices have permutable congruences, the stronger congruence-preserving extension version can be found in G. Grätzer and E. T. Schmidt [4]. We would like to point out, that Theorem 1 contains these statements. This follows from

**Lemma 9.** *The lattice $L$ of Theorem 1 is congruence permutable.*

*Proof.* This is obvious since the congruences of $L$ are congruences of $S$, which is a direct product of simple latices. □

9.3. **Deterministic isoform lattices.** Let $L$ be an isoform lattice, let $\Phi$ be a congruence of $L$, and let $A_\Phi$ be a congruence class of $\Phi$. Then any congruence class of $\Phi$ is isomorphic to $A_\Phi$. However, if $\Phi$ and $\Psi$ are congruences of $L$, it may happen that $A_\Phi$ and $A_\Psi$ as lattices are isomorphic. Let us call an isoform lattice $L$ *deterministic*, if this cannot happen; in other words, if $\Phi \neq \Psi$ are congruences of $L$, then $A_\Phi$ and $A_\Psi$ are not isomorphic.

**Lemma 10.** *The lattice $L$ of Theorem 1 can be constructed to be deterministic.*

*Proof.* The size $|A_\Phi|$ is the product of the $|S_\Phi|$, $\Phi \in \mathrm{M}(\mathrm{Con}\, K)$. Since we can easily construct the $S_\Phi$-s so that all $|S_\Phi|$-s are distinct primes, the statement follows. □

9.4. **A small construction.** Let $K$ be a finite lattice. For every meet-irreducible congruence $\Phi$ of $K$, we form the quotient lattice $K/\Phi$, and extend it to a finite, simple, separable lattice $S_\Phi$. We can construct $S_\Phi$ by adding only three elements to $K/\Phi$ as follows: $K/\Phi$ is a subdirectly irreducible lattice; so it has a covering pair $a \prec b$ such that $\mathrm{Cg}(a, b)$ is the unique minimal congruence of $K/\Phi$. Define a new element $s_\Phi$, a relative complement of $a$ in $[0, b]$, and two others: $v_\Phi^1$ and $v_\Phi^2$, new separator elements. The resulting lattice is simple.

9.5. **Rectangular vs. cubic extensions.** In G. Grätzer and E. T. Schmidt [3], we introduced a *rectangular extension* $\mathbb{R}(K)$ of a finite lattice $K$ as the direct product of all subdirect quotients $K/\Phi$. We then selected $S(K/\Phi)$, a finite, simple, sectionally complemented extension of $K/\Phi$, and formed

$$\widehat{\mathbb{R}}(K) = \prod(\, S(K/\Phi) \mid \Phi \in \mathrm{M}(\mathrm{Con}\, K)\,)$$

of $\mathbb{R}(K)$.

We changed the terminology for various reasons. (i) It seems immaterial, in general, that $S(K/\Phi)$ be sectionally complemented. (ii) $\mathbb{R}(K)$ seems to be unimportant; its congruence lattice is not really closely connected to the congruence lattice of $K$. On the other hand, the congruence lattice of $\widehat{\mathbb{R}}(K)$ is very closely connected to the congruence lattice of $K$: they have the same number of meet-irreducible elements.

9.6. **Pruned lattices.** In G. Grätzer and E. T. Schmidt [6], we introduced the concept of *pruning* a poset.

Let $Q = \langle Q; \leq_Q \rangle$ be a finite poset. Then the partial ordering $\leq_Q$ on $Q$ is the reflexive-transitive extension of $\prec_Q$, the covering relation in $\langle Q; \leq_Q \rangle$, in formula: $\mathrm{ReflTr}(\prec_Q) = \leq_Q$. Now take a subset $H$ of $\prec_Q$, and take the reflexive-transitive extension $\mathrm{ReflTr}(H)$ of $H$. Then $Q' = \langle Q; \mathrm{ReflTr}(H) \rangle$ is also a poset; we call it a *pruning* of $Q$. If you think of $Q$ in terms of its diagram, then the terminology is easy to picture: We obtain the diagram of $Q'$ from the diagram of $Q$ by cutting out (pruning) some edges (each representing a covering) but not deleting any elements.

The construction of the lattice $L$ was originally introduced by pruning the lattice $S$. While this approach may be intuitively clearer than the definition of $L$ in Section 3, it is not very practical. With the pruning definition, it is clear that we get a poset, but is is difficult to decide whether we have a lattice, and the join and meet formulas of Section 4 are very difficult to obtain.

Lemma 2 describes which edges are pruned from $S$ to give $L$. We can rephrase the description to make it easier to picture. For $\mathbf{a} \in S$ and $p \in P$, define

$$T(\mathbf{a}, p) = \{\, \mathbf{b} \in S \mid \mathbf{a}_q = \mathbf{b}_q, \text{ for all } q \neq p \,\}.$$

$T(\mathbf{a}, p)$ is a sublattice of $S$ isomorphic to $S_p$ and the covering relation in $S$ is the disjoint union of the covering relations of these sublattices. The set of edges of $T(\mathbf{a}, p)$ is either fully included in the set of edges $L$ (if there is no $q < p$ in $P$ with $a_q = v$) or it is fully pruned from $S$, that is, it is disjoint from the set of edges on $L$ (if there is $q < p$ in $P$ such that $a_q = v$).

Question: Is it true that in the construction we need only that $\mathrm{Con}\, K$ is finite (congruence finite)?

9.7. **Naturally isoform lattices.** Let $L$ be a lattice. Let us call a congruence relation $\Theta$ of $L$ *naturally isoform*, if any two congruence classes of $\Theta$ are naturally isomorphic (as lattices) in the following sense: if $a \in L$ is the smallest element of the class $a/\Theta$, then $x \mapsto x \vee a$ is an isomorphism between $0/\Theta$ and $a/\Theta$. Let us call the lattice $L$ *naturally isoform*, if all congruences of $L$ are naturally isoform.

The lattice $L$ of Theorem 1 is not naturally isoform. There is a good reason for it:

**Theorem 7.** *Let $L$ be a finite lattice. If $L$ is naturally isoform, then* $\mathrm{Con}\, L$ *is Boolean.*

*Proof.* If $L$ is simple, the statement is trivial.

Let $\Theta$ be a nontrivial congruence relation of $L$. Let $a$ be the largest element of the $\Theta$-class $0/\Theta$ and let $b$ be the smallest element of the $\Theta$-class $1/\Theta$. Then obviously $a \vee b = 1$. If $a \wedge b > 0$, then $b \vee 0 = b \vee (a \wedge b)$ $(= b)$, therefore, $x \to x \vee b$ is not an isomorphism between $0/\Theta$ and $b/\Theta$. Thus $a \wedge b = 0$.

We prove that $L \cong (a] \times (b]$. For $c \in L$, we get $c \wedge b \equiv c \wedge 1 = c$ $(\Theta)$ and so $c \wedge b \in c/\Theta$. If $d < c \wedge b$ is the smallest element of $c/\Theta$, then the natural isomorphism between $0/\Theta$ and $d/\Theta$ would force that $c \wedge b = d \vee x$, for some $0 < x \leq a$, contradicting that $(c \wedge b) \wedge a = 0$. The natural isomorphism between $0/\Theta$ and $c/\Theta$ yields that $c = (c \wedge b) \vee x$, for some unique $x \leq a$. Since $x \leq c$, clearly, $x \leq c \wedge a$. Therefore, $(c \wedge b) \vee (c \wedge a) \leq c = (c \wedge b) \vee x \leq (c \wedge b) \vee (c \wedge a)$, that is, $c = (c \wedge b) \vee (c \wedge a)$. This proves that $L \cong [0, a] \times [0, b]$.

Thus $\Theta$ is the kernel of the projection of $L$ onto $[0, a]$ and so has a complement, the kernel of the projection of $L$ onto $[0, b]$. We conclude that $\operatorname{Con} L$ is Boolean. $\square$

9.8. **Some open problems.**

**Problem 1.** *Is there an analogue of Theorem 1 for infinite lattices?*

All finite isoform lattices in [6] and in this paper are congruence permutable. So we ask:

**Problem 2.** *Is every finite isoform lattice congruence permutable?*

By [4] and also by Theorem 1 and Lemma 9 of this paper, every finite lattice has a congruence-preserving extension to a congruence permutable lattice.

**Problem 3.** *Does every lattice have a congruence-preserving extension to a congruence permutable lattice?*

By Lemma 10, every finite lattice has a congruence-preserving extension to a deterministic lattice. Can this result be extended to infinite lattices?

**Problem 4.** *Does every lattice have a congruence-preserving extension to a deterministic lattice?*

Let $L$ be a finite lattice. A congruence $\Theta$ of $L$ is *algebraically isoform* if, for every $a \in L$, there is a unary algebraic function $p(x)$ that is an isomorphism between $0/\Theta$ and $a/\Theta$. The lattice $L$ is algebraically isoform, if all congruences are algebraically isoform.

**Problem 5.** *Does every finite lattice has a congruence-preserving extension to an algebraically isoform finite lattice?*

**Problem 6.** *Can we carry out the construction for Theorem 1 in case* $\operatorname{Con} K$ *is finite (as opposed to* $K$ *is finite)?*

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