# FINITE LATTICES WITH ISOFORM CONGRUENCES

G. GRÄTZER AND E. T. SCHMIDT

Abstract. We call a lattice $L$ *isoform*, if for any congruence relation $\Theta$ of $L$, all congruence classes of $\Theta$ are isomorphic sublattices. We prove that for every finite distributive lattice $D$, there exists a finite isoform lattice $L$ such that the congruence lattice of $L$ is isomorphic to $D$.

## 1. Introduction

1.1. **The main result.** The congruence lattice of a finite lattice $L$ is characterized by a classical result of R. P. Dilworth as a finite distributive lattice $D$. Many papers were published improving this result by representing a finite distributive lattice $D$ as the congruence lattice of a finite lattice $L$ with additional properties. These results are discussed in detail—as of 1998—in Section 1.7 of Appendix A and in Section 1 of Appendix C of [1]. For a more recent survey, see G. Grätzer and E. T. Schmidt [2]. This paper is a contribution to this field, the construction of finite "isoform" lattices, a continuation of G. Grätzer, E. T. Schmidt and K. Thomsen [3], where we construct finite "uniform" lattices.

Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ *isoform*, if any two congruence classes of $\Theta$ are isomorphic (as lattices). Let us call the lattice $L$ *isoform*, if all congruences of $L$ are isoform.

**Theorem 1.** *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite isoform lattice $L$.*

Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ *uniform*, if any two congruence classes of $\Theta$ are of the same size (cardinality). Let us call the lattice $L$ *uniform*, if all congruences of $L$ are uniform. The following result was proved in G. Grätzer, E. T. Schmidt, and K. Thomsen [3]:

**Theorem.** *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite uniform lattice $L$.*

Since isomorphic lattices are of the same size, Theorem 1 is a stronger version of the theorem just cited. Figure 1 shows the result of the construction in [3] for $D = \mathbf{C}_4$, the four-element chain. This diagram shows that the lattice we obtained in the earlier paper is not isoform, although it is fairly close to it. This diagram also anticipates another property of our new construction (pruned Boolean lattice) that we are going to state in Theorem 2.

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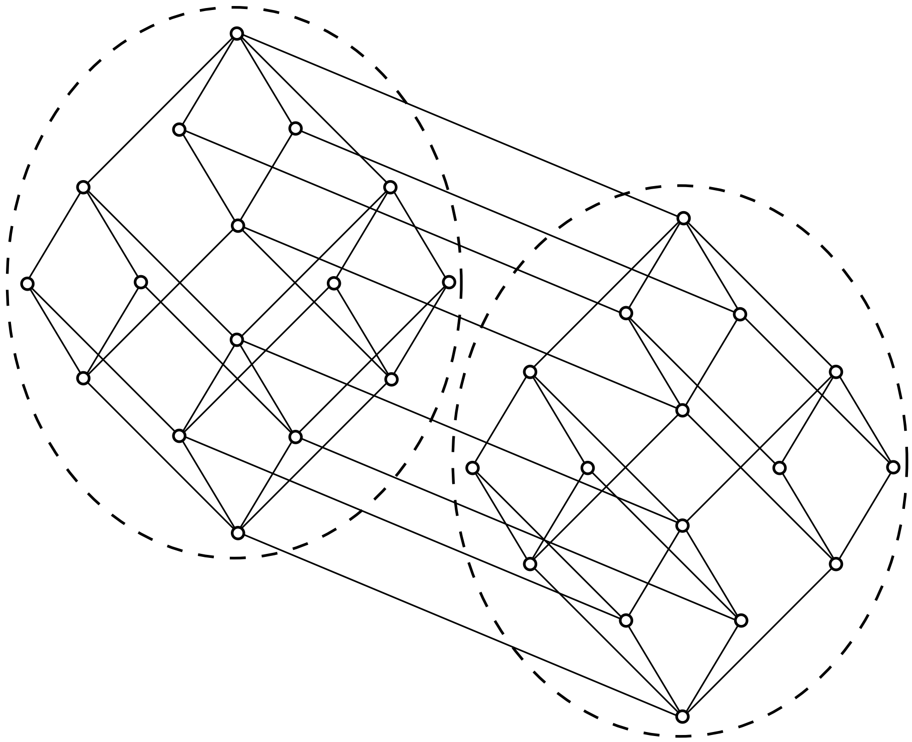

FIGURE 1. The uniform construction for the four-element chain.

Let $L$ be a lattice. Let us call the lattice $L$ *regular*, if whenever two congruences share a congruence class, then the congruences are the same. The classical proofs produce regular lattices, see [1]. Since an isoform lattice is always regular, the lattices we construct in this paper are also regular.

1.2. **Notation and concepts.** We use the standard notation, as in [1]. For a lattice $L$, we denote by $\omega_L$ and $\iota_L$ the smallest and largest congruence on $L$, respectively; we drop the subscript if $L$ is understood. If $\Theta$ is a congruence on $L$ and $[a, b]$ is an interval of $L$, we call $\Theta$ *discrete* on $[a, b]$ (or $[a, b]$ is $\Theta$-*discrete*), if $\Theta$ and $\omega$ agree on $[a, b]$, that is, $\Theta|[a, b]$ and $\omega_{[a,b]}$, where $|$ is the restriction operation. $\mathbf{C}_n$ will denote the $n$-element chain.

1.3. **The full result.** To state the main result more fully, we need two more concepts.

Let $P = \langle P; \leq_P \rangle$ be a finite poset. Then the partial ordering $\leq_P$ on $P$ is the reflexive-transitive extension of $\prec_P$, the covering relation in $\langle P; \leq_P \rangle$, in formula: $\mathrm{ReflTr}(\prec_P) = \leq_P$. Now take a subset $H$ of $\prec_P$, and take the reflexive-transitive extension $\mathrm{ReflTr}(H)$ of $H$. Then $\langle P; \mathrm{ReflTr}(H) \rangle$ is also a poset; we call it a *pruning* of $P$. If you think of $P$ in terms of its diagram, then the terminology is easy to picture: We obtain the diagram of $\langle P; \mathrm{ReflTr}(H) \rangle$ from the diagram of $P$ by cutting out (pruning) some edges (each representing a covering) but not deleting any elements. For instance, the lattice of Figure 1 is a pruning of the Boolean lattice $\mathbf{C}_2^5$.

Let $L$ be a finite lattice. We call $L$ *discrete-transitive*, if for any congruence $\Phi$ of $L$ and for $a < b < c$ in $L$, whenever $\Phi$ is discrete on $[a, b]$ and on $[b, c]$, then $\Phi$ discrete on $[a, c]$.

Here is a more complete version of our main result:

**Theorem 2.** *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite lattice $L$ with the following properties:*

(i) *$L$ is* isoform.
(ii) *For every congruence $\Theta$ of $L$, the congruence classes of $\Theta$ are projective intervals.*
(iii) *$L$ is a finite pruned Boolean lattice.*
(iv) *$L$ is discrete-transitive.*

By Properties (i) and (ii), for every congruence relation $\Theta$ of $L$ and for any two congruence classes $U$ and $V$ of $\Theta$, the congruence classes $U$ and $V$ are required to be isomorphic and projective intervals, but we do not require that there be a projectivity that is also an isomorphism. This raises the following question:

**Problem 1.** *In Theorem 2, can we require the stronger condition:*

> *For every congruence $\Theta$ of $L$, between any two congruence classes of $\Theta$, there is an isomorphism that is a projectivity.*

For a finite lattice $A$ with $|A| > 2$, a finite lattice $B$ with $|B| > 1$, and a discrete-transitive congruence $\Theta$ of $B$, we present the lattice construction $N(A, B, \Theta)$ in Section 2. The congruence structure of this new construct is described in Section 3—this is the most substantive part of the paper. The proof of Theorem 2 easily follows in Section 4 from the results in Section 3.

Anytime we prove a representation theorem of the type of Theorem 1, we raise the question whether a stronger form is available:

**Problem 2.** *Does every finite lattice have a congruence-preserving extension into a finite isoform lattice?*

Similarly, we can raise the question, what happens in the infinite case:

**Problem 3.** *Is there an analogue of Theorem 1 for infinite lattices?*

## 2. A lattice construction

Let $A$ be a nontrivial finite lattice with bounds 0 and 1; let $|A| > 2$. Set $A^- = A - \{0, 1\}$. Let $B$ be a nontrivial finite lattice with a discrete-transitive congruence $\Theta$. Note that $\iota$ is discreet-transitive.

We now introduce the lattice construction $N(A, B, \Theta)$, which was introduced in [3] in the special case $\Theta = \iota$. Note that we shall only apply this construction in the special case when $A = \mathbf{C}_2^2$, the four-element Boolean lattice.

For $u \in A \times B$, we use the notation $u = \langle u_A, u_B \rangle$, where $u_A \in A$ and $u_B \in B$. We shall denote by $\leq_\times$, $\prec_\times$, $\wedge_\times$, and $\vee_\times$ the partial ordering, the covering relation, the meet, and the join on $A \times B$, respectively. Let $B_* = \{0\} \times B$, $B^* = \{1\} \times B$, and for $b \in B$, let $A_b = A \times \{b\}$. This notation is illustrated in Figure 2.

To prune $A \times B$, we define the set:

$$\mathrm{Prune}(A, B, \Theta) = \{\, \langle \langle a, b_1 \rangle, \langle a, b_2 \rangle \rangle \mid a \in A^-,\ b_1 \prec b_2 \text{ in } B,\ \text{and}\ \ b_1 \equiv b_2\, (\Theta) \,\}.$$

$\mathrm{Prune}(A, B, \Theta)$ is a subset of $\prec_\times$, so we can define $H = \prec_\times - \mathrm{Prune}(A, B, \Theta)$. Now we take the reflexive-transitive extension $\mathrm{ReflTr}(H)$ of $H$. The set $A \times B$ with the partial ordering $\mathrm{ReflTr}(H)$ is $N(A, B, \Theta)$. We shall denote by $\leq_{N(A,B,\Theta)}$ (or simply

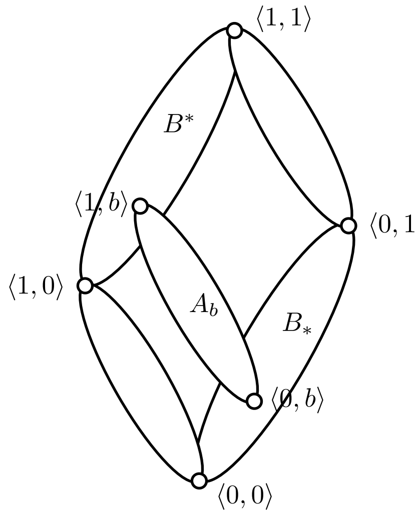


FIGURE 2. The notation for the $N(A,B,\Theta)$ construction.

by $\leq_N$, if $A$, $B$, and $\Theta$ are understood) the partial ordering of $N(A,B,\Theta)$. It is clear that if $\Theta=\omega$, then $N(A,B,\Theta)$ is the direct product $A\times B$.

We now describe $\leq_N$.

**Proposition 1.** *Let* $u,\ v\in A\times B$ *and* $u\leq_\times v$. *Then* $u\leq_N v$ *iff*

$$u_A,\ v_A\in A^- \text{ and } [u_B,v_B] \text{ is } \Theta\text{-discrete,} \tag{1}$$

*or*

$$u_A \text{ or } v_A\notin A^-. \tag{2}$$

*Proof.* Let $\leq_F$ denote the binary relation on $N(A,B,\Theta)$ defined by this proposition, that is, $u\leq_F v$ iff (1) or (2) holds.

$\leq_F$ is obviously reflexive and antisymmetric. To show that it is transitive, let $u\leq_F v$ and $v\leq_F w$. Obviously, $u\leq_\times w$. We have some cases to distinguish.

*Case 1.* Both $u\leq_F v$ and $v\leq_F w$ hold by (1). Then $u\leq_F w$ by (1), since $\Theta$ is discrete-transitive, so $[u_B,w_B]$ is $\Theta$-discrete.

*Case 2.* $u\leq_F v$ holds by (1) and $v\leq_F w$ holds by (2). So $u_A$, $v_A\in A^-$ and $v_A$ or $w_A\notin A^-$; therefore, $u_A\in A^-$ and $w_A\notin A^-$. Then $u\leq_F w$ by (2).

*Case 3.* $u\leq_F v$ holds by (2) and $v\leq_F w$ holds by (1). Then again, $u\leq_F w$ by (2), arguing as in Case 2.

*Case 4.* $u\leq_F v$ and $v\leq_F w$ both hold by (2), that is, $u_A$ or $v_A\notin A^-$ and $v_A$ or $w_A\notin A^-$. If $u_A$ or $w_A\notin A^-$, then we are done by (2). Assume, to the contrary, that $u_A$ and $w_A\in A^-$. Then $v_A\notin A^-$, so $v_A=0$ or $1$ and $u_A\leq v_A\leq w_A$. If $v_A=0$, then $u_A=0$, contradicting that $u_A\in A^-$. If $v_A=1$, then $w_A=1$, contradicting that $w_A\in A^-$.

We have proved that $\leq_F$ is a partial ordering. Finally, observe that if $u\prec v$, then $u\leq_F v$ iff $u\leq_N v$. It follows that $\leq_F\,=\,\leq_N$. $\square$

The reader may find useful the following formulation of Proposition 1:

(†) $\quad u\leq_\times v$ and $u\nleq_N v$ iff $u_A$, $v_A\in A^-$ and $[u_B,v_B]$ is not $\Theta$-discrete.

Figure 3 is a simple illustration of the $N(A, B, \Theta)$ construction with $A = \mathbf{C}_2^2$, $B = \mathbf{C}_4$, and $\Theta$ the congruence on $\mathbf{C}_4$ collapsing the two middle elements. Note that $\Theta$ is discrete-transitive.

The following lemma was proved in [3] for $\Theta = \iota$.

**Lemma 1.** *$N(A, B, \Theta)$ is a lattice under the partial ordering $\leq_N$. The meet and join in $N(A, B, \Theta)$ can be computed using the formulas:*

$$u \wedge_N v = \begin{cases} u \wedge_\times v, & \text{if } u \wedge_\times v \leq_N u \text{ and } u \wedge_\times v \leq_N v; \\ \langle 0, u_B \wedge v_B \rangle, & \text{otherwise.} \end{cases} \tag{3}$$

*and*

$$u \vee_N v = \begin{cases} u \vee_\times v, & \text{if } u \leq_N u \vee_\times v \text{ and } v \leq_N u \vee_\times v; \\ \langle 1, u_B \vee v_B \rangle, & \text{otherwise.} \end{cases} \tag{4}$$

*Proof.* Let $u$, $v \in A \times B$, and let $t$ be a lower bound of $u$ and $v$ in $N(A, B, \Theta)$.

*Case 1.* $u \wedge_\times v$ is not a lower bound of both $u$ and $v$ in $N(A, B, \Theta)$.

For instance, let $u \wedge_\times v \nleq_N u$. By (†), then

$$u_A \wedge v_A,\ u_A \in A^- \quad \text{and} \quad [u_B \wedge v_B, u_B] \text{ is not } \Theta\text{-discrete.}$$

It follows that $[t, u_B]$ is not $\Theta$-discrete, so $t \leq_N u$ implies that $t_A \notin A^-$. We cannot have $t_A = 1$, since that would imply that $u_A = 1$, contrary to $u_A \in A^-$. Therefore, $t_B = 0$, which yields that $t \leq \langle 0, u_B \wedge v_B \rangle$.

So in Case 1, $u \wedge_N v = \langle 0, u_B \wedge v_B \rangle$.

*Case 2.* $u \wedge_\times v$ is a lower bound of both $u$ and $v$ in $N(A, B, \Theta)$.

Again by (†), if $t \nleq_N u \wedge_\times v$, then $t_A$, $u_A \wedge v_A \in A^-$ and $[t, u \wedge_\times v]$ is not $\Theta$-discrete. Since $u_A \wedge v_A \in A^-$, it follows that $u_A \in A^-$ and/or $v_A \in A^-$, say, $u_A \in A^-$. Since $t \leq_N u$ and $t_A$, $u_A \in A^-$, we conclude that $[t, u]$ is $\Theta$-discrete, contradicting that $[t, u \wedge_\times v]$ is not $\Theta$-discrete.

So in Case 2, $u \wedge_N v = u \wedge_\times v$.

This verifies the meet formula. The join formula follows by duality. ☐

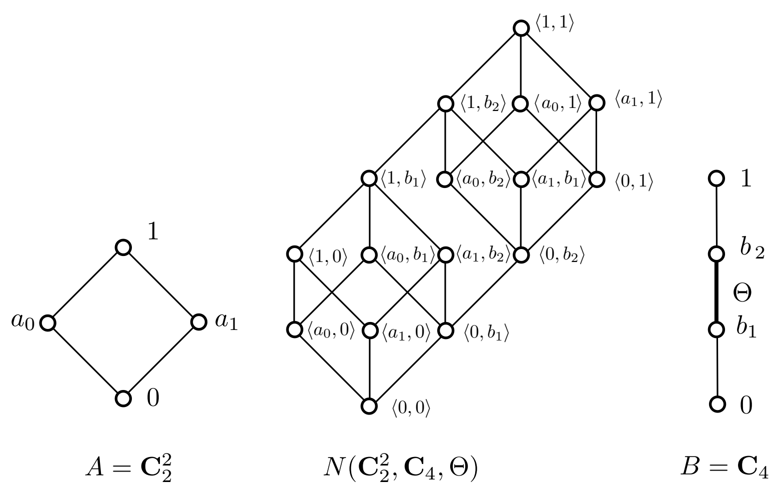


Figure 3. A simple illustration of $N(A, B, \Theta)$.

## 3. The congruences on $N(A, B, \Theta)$.

Let $A$ be a bounded lattice. A congruence $\Phi$ of $A$ *separates* 0, if $[0]\Phi = \{0\}$, that is, $x \equiv 0\ (\Phi)$ implies that $x = 0$. Similarly, a congruence $\Phi$ of $A$ *separates* 1, if $[1]\Phi = \{1\}$, that is, $x \equiv 1\ (\Phi)$ implies that $x = 1$. We call the lattice $A$ *non-separating*, if neither 0 nor 1 is separated by any congruence $\Phi \neq \omega$ of $A$.

In this section, we assume that $A$ is a non-separating finite lattice with more than two elements, $B$ is a finite lattice with more than one element, and $\Theta > \omega$ is a discrete-transitive congruence on $B$.

Let $\Psi$ be a congruence relation of $N(A, B, \Theta)$. Then define $\Psi_*$ and $\Psi^*$ as the restriction of $\Psi$ to $B_*$ and $B^*$, respectively, and define $\Psi_b$ as the restriction of $\Psi$ to $A_b$, for $b \in B$. Since $B_*$ and $B^*$ are isomorphic to $B$, we can also view $\Psi_*$ and $\Psi^*$ as congruences on $B$. Similarly, $A_b$ is isomorphic to $A$, for any $b \in B$, so we can also view $\Psi_b$ as a congruence on $A$.

We start with two easy observations.

**Lemma 2.** $\Psi_* = \Psi^*$.

*Proof.* Indeed, if $b_0 \equiv b_1\ (\Psi_*)$, then $\langle 0, b_0\rangle \equiv \langle 0, b_1\rangle\ (\Psi)$. Joining both sides with $\langle 1, 0\rangle$, we obtain that $\langle 1, b_0\rangle \equiv \langle 1, b_1\rangle\ (\Psi)$, that is, $b_0 \equiv b_1\ (\Psi^*)$. And symmetrically. □

**Lemma 3.** *The congruence $\Psi_* = \Psi^*$ of $B$ and the family of congruences*

$$\Gamma_\Psi = \{\, \Psi_b \mid b \in B \,\}$$

*of $A$ describe the congruence $\Psi$ of $N(A, B, \Theta)$.*

*Proof.* A congruence in a finite lattice is completely determined by the set of prime intervals it collapses. Since every prime interval of $N(A, B, \Theta)$ is in one of the sublattices $B_*$, $B^*$, or $A_b$, for some $b \in B$, or is perspective to a prime interval of $B_*$, the statement follows. □

We continue by analyzing $\Gamma_\Psi$ (defined in Lemma 3).

**Lemma 4.** *Either $\Gamma_\Psi = \{\, \omega_{A_b} \mid b \in B \,\}$ or $\Gamma_\Psi = \{\, \iota_{A_b} \mid b \in B \,\}$.*

*Proof.* Let us assume that $x < y \in A_b$, for some $b \in B$ and $x \equiv y\ (\Psi)$. Since $A$ is non-separating, we can assume that $x = \langle 0, x_B\rangle$, the zero of $A_b$. Joining the congruence with $\langle 0, 1\rangle$, we obtain that $\langle 0, 1\rangle \equiv \langle 0, 1\rangle \vee y\ (\Psi)$; obviously, $\langle 0, 1\rangle < \langle 0, 1\rangle \vee y$. So we can assume that $x < y \in A_1$. Again, using that $A$ is non-separating, we can assume that $x, y \in A_1$, $x \equiv y\ (\Psi)$, and $y_A = 1$.

If $x_A = 0$, then $\Psi_1 = \iota_A$, so for any $b \in B$, $\Psi_b = \iota_A$ follows be meeting the congruence $x \equiv y\ (\Psi)$ with the element $\langle 1, b\rangle$.

So let $x_A \neq 0$. By assumption, $\Theta > \omega$, so the interval $[0, 1]$ of $B$ is not $\Theta$-discrete. So if we meet the congruence $x \equiv y\ (\Psi)$ with $\langle 1, 0\rangle$, by (3), we obtain that $\langle 1, 0\rangle \equiv \langle 0, 0\rangle\ (\Psi)$, that is, $\Psi_0 = \iota_A$. So for any $b \in B$, $\Psi_b = \iota_A$ follows by joining the congruence $\langle 1, 0\rangle \equiv \langle 0, 0\rangle\ (\Psi)$ with the element $\langle 0, b\rangle$. □

Now with every congruence $\Phi$ of $B$, we associate a congruence $N(\Phi, A, B, \Theta)$ of the lattice $N(A, B, \Theta)$; we shall write $N(\Phi)$ for $N(\Phi, A, B, \Theta)$, if $A$, $B$, and $\Theta$ are understood.

**Lemma 5.** *For every congruence $\Phi$ of $B$, there exists a unique minimal congruence $N(\Phi)$ of $N(A, B, \Theta)$ satisfying $N(\Phi)_* = N(\Phi)^* = \Phi$.*

*The congruence $N(\Phi)$ of $N(A, B, \Theta)$ can be described as follows:*

$$N(\Phi) = \begin{cases} \omega_A \times \Phi, & \text{if } \Phi \wedge \Theta = \omega; \\ \iota_A \times \Phi, & \text{if } \Phi \wedge \Theta > \omega. \end{cases}$$

*Proof.* **First**, let us assume that $\Phi \wedge \Theta = \omega$. Let $\Psi = \omega_A \times \Phi$.

Obviously, $\Psi$ is an equivalence relation on $N(A, B, \Theta)$ with the property that $x \equiv y \ (\Psi)$ iff $x \wedge y \equiv x \vee y \ (\Psi)$. By Lemma I.3.8 of [1], to show that $\Psi$ is a congruence relation, it is sufficient to verify that

(SP) For $x$, $y \in N(A, B, \Theta)$ with $x < y$, and for $t \in N(A, B, \Theta)$, if $x \equiv y \ (\Psi)$, then

$$x \wedge t \equiv y \wedge t \quad (\Psi) \quad \text{and} \quad x \vee t \equiv y \vee t \quad (\Psi).$$

We now prove (SP) for meets. (SP) for joins follows dually.

So we assume that $x \equiv y \ (\Psi)$, that is,

(5) $$x_A = y_A$$

and

(6) $$x_B \equiv y_B \quad (\Phi).$$

We wish to prove that $x \wedge t \equiv y \wedge t \ (\Psi)$, that is,

(7) $$(x \wedge t)_A = (y \wedge t)_A$$

and

(8) $$(x \wedge t)_B \equiv (y \wedge t)_B \quad (\Phi).$$

By (3), the equation (8) can be rewritten as

(9) $$x_B \wedge t_B \equiv y_B \wedge t_B \quad (\Phi),$$

which always holds by (6) since $\Phi$ is a congruence on $B$. Note that by the assumption $\Phi \wedge \Theta = \omega$, (9) can be rewritten as that

(10) $$[x_B \wedge t_B, y_B \wedge t_B] \text{ is } \Theta\text{-discrete}.$$

So we wish to show that (5) and (6) imply (7).

$(y \wedge t)_A = y_A \wedge t_A$ or $(y \wedge t)_A = 0$ by (3). If $(y \wedge t)_A = 0$, then $(x \wedge t)_A \leq (y \wedge t)_A = 0$, so $(x \wedge t)_A = (y \wedge t)_A$ $(= 0)$, proving (7).

If $(y \wedge t)_A = y_A \wedge t_A$, that is, $y \wedge t = y \wedge_\times t$, then we prove that $x \wedge t = x \wedge_\times t$, which trivially verifies that (7) holds.

So assume that $y \wedge t = y \wedge_\times t$. By (3), this is equivalent to $y \wedge t \leq_N y$ and $y \wedge t \leq_N t$, which can be rewritten as follows:

One of the following conditions holds:

(11a) $$y_A \wedge t_A = 0 \text{ or } 1,$$

(11b) $$y_A = 0 \text{ or } 1,$$

(11c) $$[y_B \wedge t_B, y_B] \text{ is } \Theta\text{-discrete}.$$

and one of the following conditions holds:

(12a) $$y_A \wedge t_A = 0 \text{ or } 1,$$

(12b) $$t_A = 0 \text{ or } 1,$$

(12c) $$[y_B \wedge t_B, t_B] \text{ is } \Theta\text{-discrete},$$

We want to prove that $x \wedge t = x \wedge_{\times} t$. By (3), this is equivalent to $x \wedge t \leq_N x$ and $x \wedge t \leq_N t$, that is, one of the following conditions holds:

(13a) $$x_A \wedge t_A = 0 \text{ or } 1,$$

(13b) $$x_A = 0 \text{ or } 1,$$

(13c) $$[x_B \wedge t_B, x_B] \text{ is } \Theta\text{-discrete},$$

and one of the following conditions holds:

(14a) $$x_A \wedge t_A = 0 \text{ or } 1,$$

(14b) $$t_A = 0 \text{ or } 1,$$

(14c) $$[x_B \wedge t_B, t_B] \text{ is } \Theta\text{-discrete}.$$

We claim that more is true: (11) implies (13) and (12) implies (14).

So assume (11). Since $x_A = y_A$, (11a) implies (13a) and (11b) implies (13b). Finally, (11c) gives us that $[y_B \wedge t_B, y_B]$ is $\Theta$-discrete; by (10), $[x_B \wedge t_B, y_B \wedge t_B]$ is $\Theta$-discrete. Since $\Theta$ is discrete-transitive, we conclude that $[x_B \wedge t_B, x_B]$ is $\Theta$-discrete, verifying the conclusion of (13c).

Next assume (12) holds. Since $x_A = y_A$, (12a) implies (14a) and (12b) implies (in fact, is the same as) (14b). Finally, (12c) gives us that $[y_B \wedge t_B, t_B]$ is $\Theta$-discrete; by (10), $[x_B \wedge t_B, y_B \wedge t_B]$ is $\Theta$-discrete. Since $\Theta$ is discrete-transitive, we conclude that $[x_B \wedge t_B, t_B]$ is $\Theta$-discrete, verifying the conclusion of (14c).

**Second**, let us assume that $\Phi \wedge \Theta > \omega$. Define $\Psi = \iota_A \times \Phi$. Obviously, $\Psi$ is a congruence relation on $N(A, B, \Theta)$. Moreover, $\Psi_* = \Psi^* = \Phi$ and $\Psi_b = \iota_A$, for all $b \in B$.

Let $\Sigma$ be a congruence of $N(A, B, \Theta)$ satisfying that $\Sigma_* = \Sigma^* = \Phi$. Since $\Phi \wedge \Theta > \omega$, we can choose in $B$ the elements $b_1 \prec b_2$ such that $b_1 \equiv b_2 \ (\Phi \wedge \Theta)$. From $\Sigma_* = \Phi$, it follows that $b_1 \equiv b_2 \ (\Sigma)$ also holds. By assumption, $A$ has more than two elements, so we can choose $a \in A^-$. By (2) and (4), $\langle a, b_1 \rangle \vee \langle 0, b_2 \rangle = \langle 1, b_2 \rangle$. Since $b_1 \equiv b_2 \ (\Sigma_*)$, it follows that $\langle 0, b_1 \rangle \equiv \langle 0, b_2 \rangle \ (\Sigma)$. Joining both sides with $\langle a, b_1 \rangle$, we get that $\langle a, b_1 \rangle \equiv \langle 1, b_2 \rangle \ (\Sigma)$, and so $\langle a, b_1 \rangle \equiv \langle 1, b_1 \rangle \ (\Sigma)$. We obtain that $\Sigma_{b_1} > \omega_{A_{b_1}}$, so by Lemma 4, $\Sigma_b = \iota_{A_b}$, for all $b \in B$. We conclude that $\Sigma \geq \Psi$, so $\Psi = N(\Phi)$ is indeed the smallest congruence of $N(A, B, \Theta)$ satisfying that $\Psi_* = \Psi^* = \Phi$. □

The map $N \colon \Phi \mapsto N(\Phi)$ maps $\operatorname{Con} B$ into $\operatorname{Con} N(A, B, \Theta)$. This map has many interesting properties.

**Lemma 6.**

(i) *The map* $N \colon \Phi \mapsto N(\Phi)$ *is an order preserving, one-to-one map of* $\operatorname{Con} B$ *into* $\operatorname{Con} N(A, B, \Theta)$.

(ii) *The map* $N$ *is an order preserving, one-to-one map of the join-irreducible elements of* $\operatorname{Con} B$ *into the join-irreducible elements of* $\operatorname{Con} N(A, B, \Theta)$.

(iii) *The lattice* $\operatorname{Con} N(A, B, \Theta)$ *has exactly one join-irreducible element that is not in the image of* $N$*:*

$$\Sigma = \Theta(\langle 0, 0 \rangle, \langle 1, 0 \rangle).$$

$\Sigma$ *is a minimal join-irreducible element of* $\operatorname{Con} N(A, B, \Theta)$.

(iv) *For a minimal join-irreducible congruence* $\Phi$ *of* $B$*, we have*

$$\Sigma < N(\Phi) \quad \textit{iff} \quad \Phi \leq \Theta.$$

*Proof.* Statement (i) follows directly from Lemma 5.

A join-irreducible congruence of a finite lattice is one that is generated by a covering pair of elements. If $\Phi = \Theta(b_1, b_2)$ with $b_1 \prec b_2$ in $B$, then $N(\Phi) = \Theta(\langle 0, b_1\rangle, \langle 0, b_2\rangle)$ and $\langle 0, b_1\rangle \prec \langle 0, b_2\rangle$ in $N(A, B, \Theta)$. So the join-irreducible congruences of $B$ are mapped by $N$ into join-irreducible congruences of $N(A, B, \Theta)$, verifying (ii).

As we argued in the proof of Lemma 3, any prime interval of $N(A, B, \Theta)$ is in one of the sublattices $B_*$, $B^*$, or $A_b$, for some $b \in B$, or is perspective to a prime interval of $B_*$. The prime intervals in $B_*$ and $B^*$ generate the join-irreducible congruences of the form $N(\Phi)$, where $\Phi$ is a join-irreducible congruence of $B$. The remaining prime intervals all generate the same join-irreducible congruence, $\Sigma$, by Lemma 4, verifying (iii).

$\Sigma < N(\Phi)$ holds iff $\Phi \wedge \Theta > \omega$. If $\Phi \wedge \Theta < \Phi$, then there is a join-irreducible congruence of $B$ below $\Phi \wedge \Theta$, so properly below $\Phi$, contrary to assumption. Therefore, $\Phi \wedge \Theta = \Phi$, that is, $\Phi \leq \Theta$, verifying (iv). □

Let $D$ be a finite distributive lattice. Let $\mathrm{J}(D)$ denote the poset of join-irreducible elements of $D$. For a minimal join-irreducible element $p$ of $D$, let $\mathrm{Cov}(p)$ denote the covers of $p$ in $\mathrm{J}(D)$, that is, the set of all join-irreducible elements $q$ of $D$ for which $p \prec q$ in $\mathrm{J}(D)$. Let $D'$ denote the join-subsemilattice of $D$ generated by $\mathrm{J}(D) - \{p\}$. Obviously, $D'$ is a finite distributive lattice with $\mathrm{J}(D') = \mathrm{J}(D) - \{p\}$. The set $\mathrm{Cov}(p)$ is an antichain of $\mathrm{J}(D')$.

Conversely, given a finite distributive lattice $D'$ and an antichain $C \neq \varnothing$ of $\mathrm{J}(D')$, we can form the poset $\mathrm{J}(D') \cup \{p\}$, where $p \notin \mathrm{J}(D')$, and we can extend the partial ordering of $\mathrm{J}(D')$ to $\mathrm{J}(D') \cup \{p\}$ by requiring that $p < q$, for all $q \in C$; more precisely, we must set $p < r$, for every $r \in \mathrm{J}(D')$ for which there exists a $q \in C$ satisfying $q \leq r$. The poset $\mathrm{J}(D') \cup \{p\}$ determines a distributive lattice $D$. Obviously, in $D$, we have $\mathrm{Cov}(p) = C$.

We call $D'$ the distributive lattice we obtain from $D$ by *deleting the minimal join-irreducible element* $p$, and we call $D$ the distributive lattice obtained from $D'$ by *adding a minimal join-irreducible element under* $C$.

Now we summarize what we have learned about the congruence lattice of our construct $N(A, B, \Theta)$.

**Theorem 3.** *Let $A$ be a finite non-separating lattice with more than two elements. Let $B$ be a finite lattice with more than one element, and let $\Theta > \omega$ be a discrete-transitive congruence on $B$. Let $\Theta = \Sigma_1 \vee \cdots \vee \Sigma_n$ be an irredundant representation of $\Theta$ as a join of join-irreducible elements, and set $C = \{\Sigma_1, \ldots, \Sigma_n\}$. Finally, let $\Sigma$ be the join-irreducible congruence of $N(A, B, \Theta)$ defined in Lemma 6: $\Sigma = \Theta(\langle 0, 0\rangle, \langle 1, 0\rangle)$.*

*Then we can obtain—up to isomorphism—the congruence lattice of $N(A, B, \Theta)$ by adjoining to the congruence lattice of $B$ a minimal join-irreducible element under $C$.*

*Equivalently, we can obtain—up to isomorphism—the congruence lattice of $B$ by deleting the minimal join-irreducible element $\Sigma$ of* $\mathrm{Con}\, N(A, B, \Theta)$.

We need one more property of our construction.

**Lemma 7.** *Let $A$, $B$, and $\Theta$ be given as in Theorem 3. If the congruence $\Phi$ of $B$ is discrete-transitive, then so is $N(\Phi)$.*

*Proof.* **First**, let us assume that $\Phi \wedge \Theta = \omega$, so $N(\Phi) = \omega_A \times \Phi$ by Lemma 5. For elements $a < b \in N(\Phi)$, then $a \equiv b\ (N(\Phi))$ iff $a_A = b_A$ and $a_B \equiv b_B\ (\Phi)$. It follows that an interval $[u, v]$ of $N(A, B, \Theta)$ is $N(\Phi)$-discrete iff the interval $[u_B, v_B]$ of $B$ is $\Phi$-discrete. This clearly implies that if $\Phi$ is discrete-transitive in $B$, then $N(\Phi)$ is discrete-transitive in $N(A, B, \Theta)$.

**Second**, let us assume that $\Phi \wedge \Theta > \omega$, so $N(\Phi) = \iota_A \times \Phi$ by Lemma 5. Then an interval $[u, v]$ of $N(A, B, \Theta)$ is $N(\Phi)$-discrete iff $u_A = v_A$ and the interval $[u_B, v_B]$ of $B$ is $\Phi$-discrete. The discrete-transitivity of $N(\Phi)$ follows. □

**Corollary.** *Let $A$, $B$, and $\Theta$ be given as in Theorem 3. If all congruence of $B$ are discrete-transitive, then all congruence of $N(A, B, \Theta)$ are discrete-transitive.*

*Proof.* First, observe that $\Sigma = \Theta(\langle 0, 0\rangle, \langle 1, 0\rangle)$ is discrete-transitive. Now the statement follows because all the congruences of $N(A, B, \Theta)$ that have not yet been proven discrete-transitive are of the form $\Sigma \vee N(\Phi)$, and the join of two discrete-transitive congruences is clearly discrete-transitive. □

## 4. The proof of Theorem 2

Let $D$ be a finite distributive lattice. We have to construct a lattice $L$ satisfying the requirements of Theorem 2.

If $D$ is the one-element lattice, then let $L$ be the one-element lattice.

If $D$ has more than one element, then $\mathrm{J}(D) \neq \varnothing$, and we use induction on the size of $\mathrm{J}(D)$.

If $|\,\mathrm{J}(D)| = 1$, then let $L = \mathbf{C}_2$. If $|\,\mathrm{J}(D)| = 2$, then either $\mathrm{J}(D)$ is unordered and then let $L = \mathbf{C}_2^2$, or $J(D)$ is the two-element chain, in this case let $L$ be the lattice of Figure 4. Obviously, the lattice of Figure 4 satisfies the conditions of Theorem 2.

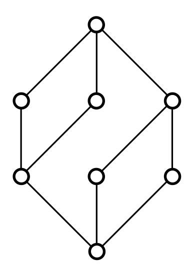

Figure 4. The lattice $L$ for the three-element chain.

Now for the induction step, let $|\mathrm{J}(D)| = n > 2$. If $\mathrm{J}(D)$ is an antichain, the statement is trivial; for $L$ we can choose a finite Boolean lattice. Otherwise, choose a minimal but not maximal join-irreducible element $p$ of $D$. Let $D'$ be the distributive lattice join-generated by $J(D) - \{p\}$. Then $|\,\mathrm{J}(D')| = n - 1$, so by the induction hypothesis, there is a lattice $B$ and a lattice isomorphism $\alpha\colon D' \to \operatorname{Con} B$ satisfying Conditions (i)–(iv) of Theorem 2.

Since $p$ is not a maximal element of $\mathrm{J}(D)$, it follows that $\operatorname{Cov}(p) \neq \varnothing$. Take a $q \in \operatorname{Cov}(p)$. Since $q \in D'$, under the isomorphism $\alpha$ it is mapped to a congruence $\Theta_q$ of $B$. Now we define

$$\Theta = \bigvee (\, \Theta_q \mid q \in \operatorname{Cov}(p)\,).$$

The lattice $L$ of Theorem 2 is defined as

$$L = N(\mathbf{C}_2^2, B, \Theta).$$

By Theorem 3, we have the isomorphism $\operatorname{Con} L \cong D$.

We have to prove that the lattice $L$ satisfies Conditions (i)-(iv) of Theorem 2.

To investigate the congruence classes of $L$, by Theorem 3 (see Lemmas 5 and 6 for more detail), a congruence $\Psi$ of $L$ is one of the following form:

*Form 1.* $\Psi = N(\Phi) = \omega_A \times \Phi$, where $\Phi$ is a congruence of $B$ satisfying $\Phi \wedge \Theta = \omega$.

*Form 2.* $\Psi = N(\Phi) = \iota_A \times \Phi$, where $\Phi$ is a congruence of $B$ satisfying $\Phi \wedge \Theta > \omega$.

*Form 3.* $\Psi = N(\Phi) \vee \Sigma$, where $\Phi$ is a congruence of $B$.

If $\Psi$ is of Form 1, then the congruence classes of $N(\Phi)$ are described as follows: Let $[u, v]$ be a congruence class of $\Phi$ in $B$. Then the congruence classes of $\Psi$ in $L$ are exactly the intervals of the form $[\langle a, u\rangle, \langle a, v\rangle]$, for any $a \in A$. Obviously, from (2) and Lemma 1, the interval $[u, v]$ of $B$ is isomorphic to the interval $[\langle a, u\rangle, \langle a, v\rangle]$ of $L$.

Now if $[u, v]$ and $[u', v']$ are any two congruence classes of $\Phi$ in $B$, then $[u, v]$ and $[u', v']$ are isomorphic intervals and they are projective, by induction hypothesis. It is obvious, then, that $[\langle a, u\rangle, \langle a, v\rangle]$ and $[\langle a', u'\rangle, \langle a', v'\rangle]$ are isomorphic, for any $a$, $a' \in A$.

We also have to show that $[\langle a, u\rangle, \langle a, v\rangle]$ and $[\langle a', u'\rangle, \langle a', v'\rangle]$ are projective. Since $[\langle a, u\rangle, \langle a, v\rangle]$ is perspective to $[\langle 0, u\rangle, \langle 0, v\rangle]$ and $[\langle a', u'\rangle, \langle a', v'\rangle]$ is perspective to $[\langle 0, u'\rangle, \langle 0, v'\rangle]$, it is sufficient to show that $[\langle 0, u\rangle, \langle 0, v\rangle]$ and $[\langle 0, u'\rangle, \langle 0, v'\rangle]$ are projective.

By the induction hypothesis, $[u, v]$ and $[u', v']$ are projective. A trivial induction shows (see Section III.1 of [1]) that it is sufficient to verify that if $[u, v]$ and $[u', v']$ are perspective, then so are $[\langle 0, u\rangle, \langle 0, v\rangle]$ and $[\langle 0, u'\rangle, \langle 0, v'\rangle]$. By duality, it is sufficient to compute this for "up" perspectivity: So let $v \wedge u' = u$ and $v \vee u' = v'$. Then obviously, $\langle 0, v\rangle \wedge \langle 0, u'\rangle = \langle 0, u\rangle$ and $\langle 0, v\rangle \vee \langle 0, u'\rangle = \langle 0, u\rangle$, completing the case.

If $\Psi$ is of Form 2 or 3, then the congruence classes of $\Psi$ are described in Lemmas 5 and 6 as follows: Let $[u, v]$ be a congruence class of $\Phi$ in $B$. Then the congruence classes of $\Psi$ in $L$ are exactly the intervals of $L$ of the form $[\langle 0, u\rangle, \langle 1, v\rangle]$. Now observe that $[\langle 0, u\rangle, \langle 1, v\rangle]$ is isomorphic to $N(\mathbf{C}_2^2, [u, v], \iota_{[u,v]})$, so if the intervals $[u, v]$ and $[u', v']$ of $B$ are isomorphic, so are the intervals $[\langle 0, u\rangle, \langle 1, v\rangle]$ and $[\langle 0, u'\rangle, \langle 1, v'\rangle]$ of $L$.

Then we have to show that any two congruence classes of $\Psi$ are projective intervals. So let $[u, v]$ and $[u', v']$ be any two congruence classes of $\Phi$ in $B$. Then $[\langle 0, u\rangle, \langle 1, v\rangle]$ and $[\langle 0, u'\rangle, \langle 1, v'\rangle]$ are the corresponding $\Psi$ classes in $L$. Now if $[u, v]$ is "up" perspective to $[u', v']$, that is, $v \vee u' = v'$ and $v \wedge u' = u$, then obviously $\langle 1, v\rangle \vee \langle 0, u'\rangle = \langle 1, v'\rangle$ and $\langle 1, v\rangle \wedge \langle 0, u'\rangle = \langle 0, u\rangle$. By duality, we get "down" perspectivity, and by induction, projectivity.

This completes the proof of Conditions (i) and (ii). Condition (iii) is obvious: By induction hypothesis, $B$ is a pruned Boolean lattice. Of course, $\mathbf{C}_2^2$ is a Boolean lattice. So $L$ is a pruned Boolean lattice. Finally, by the Corollary to Lemma 7, the congruences of $L$ are discrete-transitive.

Note that to prove Theorem 1, any non-separating finite lattice $A$ with more than two elements will do. To prove Theorem 2, we can choose $A$ as any non-separating finite lattice with more than two elements, provided that $A$ is a pruned, finite Boolean lattice. In either case, the smallest lattice we can choose is $A = \mathbf{C}_2^2$.

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