THE $M_3[D]$ CONSTRUCTION

Let D be a bounded distributive lattice, and let $M_3 = \{0, a, b, c, 1\}$ be the five-element non-distributive modular lattice. Then $M_3[D]$ denote the subposet of D^3 consisting of all $< x, y, z >$ satisfying $x \wedge y = y \wedge z = z \wedge x$. We call such a triple *balanced*. This was introduced in [5].

Theorem. *$M_3[D]$ satisfies the following conditions:*

(i) *$M_3[D]$ is a modular lattice.*

(ii) *The subset $\overline{M}_3 = \{< 0,0,0 >, < 1,0,0 >, < 0,1,0 >, < 0,0,1 >, < 1,1,1 >\}$ of $M_3[D]$ is a sublattice of $M_3[D]$ and it is isomorphic to M_3,*

(iii) *The subposet $\overline{D} = \{< x,0,0 >\} x \in D$ of $M_3[D]$ is isomorphic to D; we identify $\overline{D}$ with D,*

(iv) *$\overline{M}_3$ and D generate $M_3[D]$,*

(v) *Let Θ be a congruence relation of $\overline{D} = D$; then there is a unique congruence $\overline{\Theta}$ of $M_3[D]$ such that $\overline{\Theta}$ restricted to D is Θ ; therefore,* $\mathrm{Con}\, M_3[D] \cong \mathrm{Con}\, D$.

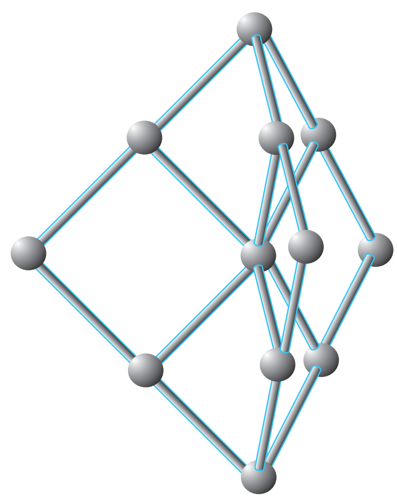

Figure 1. $M_3[D]$ where D is the three-element chain

In today's terminology, $M_3[D]$ is a congruence preserving extension of D. The extension $M_3[D]$ is shown in Figure 1., where D is the three-element chain (see a rotary version).

Remarks. Let M_n be the modular but not distributive lattice with n atoms. In this case we can define $M_n[D]$ similarly. It is easy to see that $M_3[D]$ is a special subdirect power of M_3, it is the lattice of all order-preserving mappings $f : \mathrm{J}(D) \longrightarrow M_3$ (don't forget $\{f : \mathrm{J}(D) \longrightarrow \mathbf{2}\} \cong \mathbf{D}$, where $\mathrm{J}(D)$ is the poset of all join-irreducible elements of D and $\mathbf{2}$ is the two-element lattice).

Date: February 3, 2008.

The first important generalization of this construction was the *Boolean triple construction* [2] which is a special case of a more general lattice tensor product construction of G. Grätzer and F. Wehrung (see [2]). These constructions play an important role by the congruence-preserving extensions. See more in Grätzer's new book [3]

Other interesting generalizations, related results in: R. W. Quackenbush [4], J. D. Farley [1].

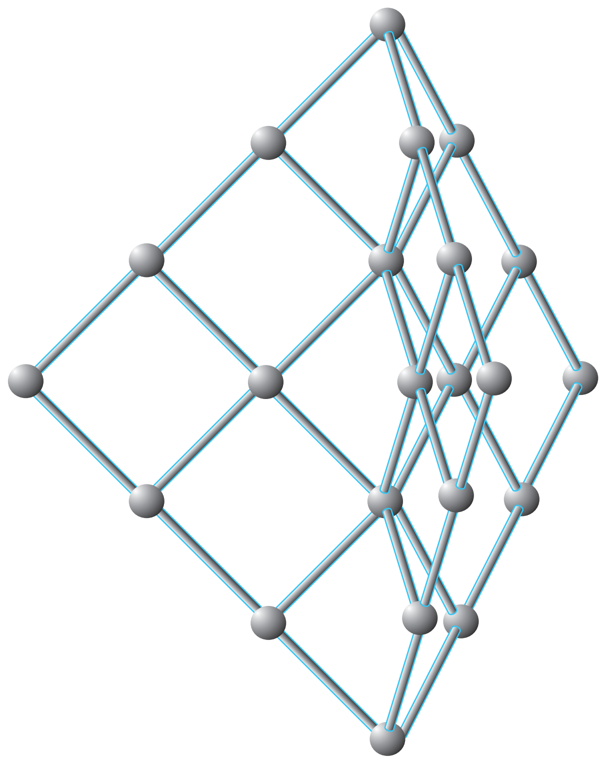

FIGURE 2. D is the four-element chain

REFERENCES

[1] J. D. Farley, *Priestley powers of lattices and their congruences: a problem of E. T. Schmidt*, Acta Sci. Math. 62 (1996), 3-45

[2] G. Grätzer, F. Wehrung, *Proper congruence-preserving extensions of lattices*, Acta Math. Hungar. 85 (1999), 175-185).

[3] G. Grätzer, *The congruences of finite lattices*, Birkhäuser, 2006

[4] R. W. Quackenbush, *Non-modular varieties of semimodular lattices with a spanning* M_3, Discrete Math. 53 (1985), 186-196

[5] E. T. Schmidt *Zur Characterisierung der Kongruenzverbände der Verbände*, Math. Casopis Sloven. Acad. Vied. 18 (1968), 3-20.