Conference honoring 5×80 of János Aczél, Ákos Császár, László Fuchs, István Gál and János Horváth

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László Fuchs was my teacher who introduced me to the algebra

My lecture

Congruence lattices

Grätzer-Schmidt, 1963:

- **Theorem 1.** *Let L be an algebraic lattice. Then there exists an algebra A whose congruence lattice is isomorphic to L.*

- It is perhaps the most famous open problem of universal algebra whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. Pálfy-Pudlak proved: it is equvivalant to a group theoretical question:

- **Problem.** *Given a finite lattice L, do there exist a finite group G and a subgroup H such that the interval [H, G] in the subgroup lattice of G is isomorphic to L?*

Complete congruences

- For complete lattices we have complete congruences, and the complete lattice of complete congruences. These lattices were characterized by G. Grätzer:
- **Theorem 2.** *Every complete lattice K can be represented as the lattice of complete congruence relations of a complete lattice L.*

In a series of papers, much sharper results have been obtained, culminating in Grätzer-Schmidt, 1993:

- **Theorem 3.** *Every complete lattice L can be represented as the lattice of complete congruence relations of a complete distributive lattice D.*

Congruence lattices of lattices

- For every lattice *L* it is clear that the congruence **Con**(*L*) is algebraic. By a result of Nakayama and Funayama **Con**(*L*) is also distributive. Is the converse true: is every distributive algebraic lattice isomorphic to the congruence lattice of a suitable lattice ? This is one of the most famous open question of the lattice theory.
- It is more convenient to consider **Comp**(*L*), the distributive semilattice of compact congruences of the lattice *L*. The original question can be rephrased: is every distributive semilattice *S* isomorphic to the semilattice of all compact congruences of a lattice *L* ? In tis case we say *S* is representable.
- Each one of the following conditions implies that *S* is representable:

The sufficient conditions:

- *S is a lattice* (E. T. Schmidt, 1968; see P. Pudlak, 1985),
- *S is locally countable* (that is for every *s* in *S,* (*s*] is countable, A. P. Huhn 1983, H. Dobbertin),
- |*S*| ≤ $\aleph_1$ (A. P. Huhn).

It was hoped for a long time that the two sucsessful approaches solving the case for a lattice *S* can be used to answer the general question.

F. Wehrung proved that neither method can answer the general question even the lattices of size $\aleph_2$

Lattices with nice congruences

Dilworth theorem: *every finite distributive lattice D is isomorphic to the congruence lattice if a finite lattice.*

We want:

Every finite distributive lattice D can be represented as the congruence lattice of a nice finite lattice

We have a sequence of such theorems:

The poset of join-irreducible elements

- A finite distributive lattice *D* is determined by the poset **J**(*D*) of join-irreducible elements. So a representation of a finite distributive lattice *D* as the congruence lattice of a lattice *L* is really a representation of a finite poset *P* (= **J**(*D*)) as the poset of join-irreducible congruences of a finite lattice *L.*

We want:

- *Every finite poset P can be represented as the poset of join-irreducible congruences of a nice finite lattice L.*

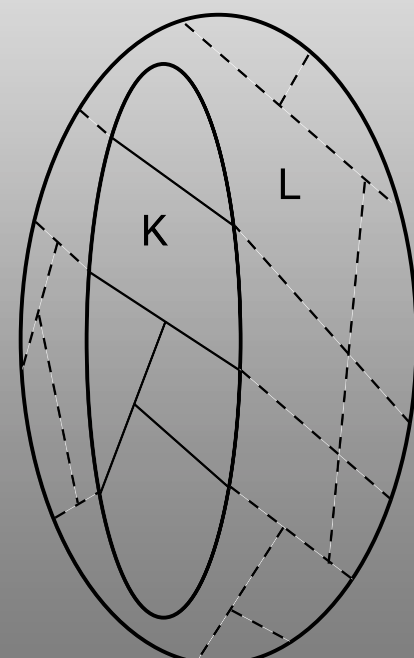
Two typs of representations theorems:

- The straight representation theorems
- The *congruence-preserving extension* results.

Let K be a finite lattice. A finite lattice L is a *congruence-preserving* of K, if L is an extension and every congruence Θ of K has exactly one extension Φ to L – that is $\Phi|K = \Theta$.

Of course, the congruence lattice of K is isomorphic to the congruence lattice of L. See the next figure.

Congruence-preserving extension



Nice = sectionally complemented

- **Theorem 4.** (G. Grätzer and E. T. Schmidt, 1962)
  Every finite distributive lattice D can be represented as the congruence lattice of a finite sectionally complemented lattice L.

- **Theorem 5.** (G. Grätzer and E. T. Schmidt, 1999)
  Every finite lattice K has a finite, sectionally-complemented, congruence-preserving extension L.

Nice = minimal

The lattice L constructed by R. P. Dilworth to represent D is very large, it has $O(2^{2n})$ elements

- **Theorem 6.** (G. Grätzer, H. Lakser and E. T. Schmidt 1996). *Let D be a finite distributive lattice with n join-irreducible elements. Then there exists a planar lattice L of $O(n^2)$ elements with* **Con**$(L) \approx D$.

Nice = semimodular

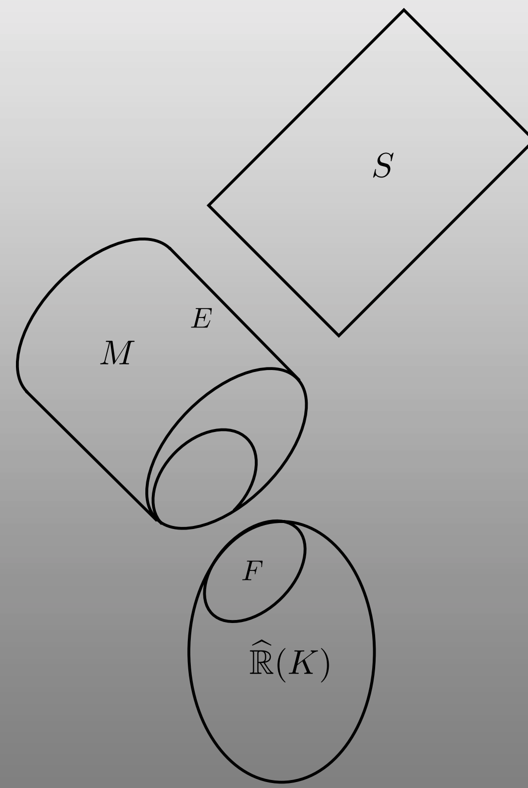
- **Theorem 7.** (G. Grätzer, H. Lakser and E. T. Schmidt, 1998). *Every finite distributive lattice D can be represented as the congruence lattice of a finite semimodular lattice S. In fact, S can be constructed as a planar lattice of size $O(n^3)$, where n is the number of join-irreducible elements of D*

- **Theorem 8.** (G. Grätzer and E. T. Schmidt, 2001). *Every finite lattice K has a congruence-preserving embedding into a finite semimodular lattice L.*

Nice = semimodular

The proof starts out with the cubic extension R(K) of K, where we choose each $S(K_i)$ semimodular. So the cubic extension is semimodular. The congruences then are represented in a dual ideal F of R(K) that is Boolean. By gluing a suitable modular lattice M to R(K). The congruences are then represented on a dual ideal E' of M that is a chain, so the proof is completed by gluing the lattice S to the construct:

Semimodular construction



Nice = given authomorphism group

- **Theorem 9.** (The independence theorem, V. A. Baranskii and A. Urquhart, 1979). *Let D be a finite distributive lattice with more than one element, and let G be a finite group. Then there exists a finite lattice L such that the congruence lattice of L is isomorphic to D and the automorphism group of L is isomorphic to G.*

This is a representation theorem. There is also a congruence-preserving extension variant for this result.

Strong independence theorem

- **Theorem 10.** (G. Grätzer and E. T. Schmidt, 1995). *Let K be a finite lattice with more then one element and let G be a finite group. Then K has a congruence-preserving extension L whose automorphism group is isomorphic to G.*

Nice = regular

Let *L* a lattice. We call a congruence relation Θ *regular*, if any congruence class of Θ determines the congruences. Let us call the lattice *L regular*, if all congruences of *L* are regular. Sectionally complemented lattices are regular, so we alredy have a representation theorem (Theorem 4).

- **Theorem 11.** *Every finite lattice L has a congruence-preserving embedding into a finite regular lattice*

We have this theorem for arbitary infinite lattice (Grätzer and Schmidt, 2001):

- **Theorem 12.** *Every lattice has a congruence-preserving embedding into a regular lattice.*

Nice = uniform

Let *L* be a lattice. We call a congruence relation Θ of *L uniform*, if ant two congruence classes of Θ are of the same size (cardinality). Let us call the lattice *L uniform*, if all congruences of *L* are uniform.

- **Theorem 13.** (G. Grätzer, E. T. Schmidt and K. Thomsen, 2002). *Every finite distributive lattice D can be represented as the congruence lattice of a finite uniform lattice L.*

A uniform lattice is always regular, so the lattice *L* of this theorem is also regular.

Nice = isoform

Let *L* be a lattice. We call a congruence relation Θ of *L* *isoform*, if any two congruence classes of Θ are isomorphic (as lattices). Let us call the lattice *isoform*, if all congruences of *L* are isoform.

- **Theorem 14.** (G. Grätzer and E. T. Schmidt, 2002). *Every finite distributive lattice D can be represented as the congruence lattice of a finite, isoform lattice L.*

- **Theorem 15.** (G. Grätzer, R. W. Quackenbush and E. T. Schmidt, 2004). *Every finite lattice K has a congruence-preserving extension to a finite isoform lattice L.*

Simultaneous representations

Let *L* be a lattice and let *K* be a sublattice of *L*. Then the restriction map **rs**: **Con** *L* **Con** *K* is a {0,1} preserving meet-homomorphism.

G. Grätzer and H. Lakser, 1986:

- **Theorem 16.** *Let D and E be finite distributive lattices, let D have more then one element. Let φ be a {0,1}-homomorphism of D into E. Then there exists a (sectionally complemented) finite lattice L and an ideal K of L such that D ≈* **Con** *L, E ≈* **Con** *K, and φ is represented by* **rs**, *the restriction map.*

Open questions:

- **Problem 1.** *Let D and E be finite distributive lattices; let D have more than one element. Let φ be a {0,1}-homomorphism of D into E. Does there exists a finite isoform lattice L and an isoform ideal K of L such that D ≈* **Con** *L, E ≈* **Con** *K, and φ is represented by the restriction map ?*
- **Problem 2.** *Is every finite distributive lattice D isomorphic to the congruence lattice of an isoform modular lattice ?*