MATH302 Formula Sheet for Final Exam on August 13, 2014

Discrete random variables:

Bernoulli, $\text{Ber}(p)$:

$$\mathbf{P}(X=1)=p,\quad \mathbf{P}(X=0)=1-p,\quad \mathbf{E}(X)=p,\quad \mathbf{Var}(X)=p(1-p)$$

Binomial, $\text{Bin}(n,p)$:

$$\mathbf{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}\mathbb{1}[0\le k\le n],\quad \mathbf{E}(X)=np,\quad \mathbf{Var}(X)=np(1-p)$$

Geometric, $\text{Geo}(p)$:

$$\mathbf{P}(X=k)=p(1-p)^{k-1}\mathbb{1}[1\le k],\quad \mathbf{E}(X)=\frac{1}{p},\quad \mathbf{Var}(X)=\frac{1-p}{p^2}$$

Negative binomial, $\text{NBin}(r,p)$:

$$\mathbf{P}(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}\mathbb{1}[r\le k],\quad \mathbf{E}(X)=\frac{r}{p},\quad \mathbf{Var}(X)=r\frac{1-p}{p^2}$$

Poisson, $\text{Poi}(\lambda)$:

$$\mathbf{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}\mathbb{1}[k\ge 0],\quad \mathbf{E}(X)=\lambda,\quad \mathbf{Var}(X)=\lambda$$

Continuous random variables:

Uniform, $\text{Unif}[a,b]$:

$$f(x)=\frac{1}{b-a}\mathbb{1}[a\le x\le b],\quad \mathbf{E}(X)=\frac{a+b}{2},\quad \mathbf{Var}(X)=\frac{(b-a)^2}{12}$$

Exponential, $\text{Exp}(\lambda)$:

$$f(x)=\lambda e^{-\lambda}\mathbb{1}[x\ge 0],\quad \mathbf{E}(X)=\frac{1}{\lambda},\quad \mathbf{Var}(X)=\frac{1}{\lambda^2}$$

Normal, $\mathcal{N}(\mu,\sigma)$:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\quad \mathbf{E}(X)=\mu,\quad \mathbf{Var}(X)=\sigma^2$$

Basic rules of probability:

Tower rule:

$$\mathbf{P}(E_1\cap E_2\cap\cdots\cap E_n)=\mathbf{P}(E_1)\mathbf{P}(E_2\,|\,E_1)\ldots\mathbf{P}(E_n\,|\,E_1\cap\cdots\cap E_{n-1})$$

Total probability rule:
If $F_1,F_2,\ldots,F_n$ are mutually exclusive and $F_1\cup F_2\cup\cdots\cup F_n=\Omega$ then

$$\mathbf{P}(E)=\sum_{k=1}^{n}\mathbf{P}(E\,|\,F_k)\mathbf{P}(F_k)$$

Bayes' rule:

$$\mathbf{P}(E\,|\,F)=\frac{\mathbf{P}(F\,|\,E)\mathbf{P}(E)}{\mathbf{P}(F)}$$

Expectation, Variance:

$$\mathbf{E}\big(\,g(X)\,\big) = \sum_{x\in\mathcal{I}} g(x)f(x), \qquad \mathbf{E}\big(\,g(X)\,\big) = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x$$

$$\mathbf{E}\big(\,g(X,Y)\,\big) = \sum_{x\in\mathcal{I}}\sum_{y\in\mathcal{J}} g(x,y)f(x,y), \qquad \mathbf{E}\big(\,g(X,Y)\,\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\mathbf{Var}\big(\,X\,\big) = \mathbf{E}\big(\,\big(X-\mathbf{E}\big(\,X\,\big)\big)^2\,\big) = \mathbf{E}\big(\,X^2\,\big) - \mathbf{E}\big(\,X\,\big)^2$$

$$\text{Chebychev's inequality:} \qquad \mathbf{P}\big(\,|X-\mu| \geq \varepsilon\,\big) \leq \frac{\sigma^2}{\varepsilon^2}$$

$$\mathbf{E}\big(\,aX+b\,\big) = a\mathbf{E}\big(\,X\,\big) + b, \quad \mathbf{E}\big(\,X+Y\,\big) = \mathbf{E}\big(\,X\,\big) + \mathbf{E}\big(\,Y\,\big)$$

$$\mathbf{Var}\big(\,aX+b\,\big) = a^2\mathbf{Var}\big(\,X\,\big), \quad \mathbf{Var}\big(\,X+Y\,\big) \overset{\text{indep}}{=} \mathbf{Var}\big(\,X\,\big) + \mathbf{Var}\big(\,Y\,\big)$$

Cumulative distribution functions and probability density functions:

$$\text{c.d.f.:} \quad a \leq b \implies F(a) \leq F(b), \quad \lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to\infty} F(x) = 1$$

$$\text{p.d.f.:} \quad f(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x), \qquad F(a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x$$

$$f_X(x) = \sum_{y\in\mathcal{J}} f(x,y), \qquad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$$

$$\mathbf{P}\big(\,(X,Y)\in A\,\big) = \iint_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

If (X,Y) are independent and $Z = X+Y$ then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,\mathrm{d}y$$

Covariance:

$$\mathbf{Cov}\big(\,X\,,\,Y\,\big) = \mathbf{E}\big(\,\big(X-\mathbf{E}\big(\,X\,\big)\big)\big(Y-\mathbf{E}\big(\,Y\,\big)\big)\,\big) = \mathbf{E}\big(\,XY\,\big) - \mathbf{E}\big(\,X\,\big)\mathbf{E}\big(\,Y\,\big)$$

$$\mathbf{Cov}\big(\,X\,,\,X\,\big) = \mathbf{Var}\big(\,X\,\big), \quad \mathbf{Cov}\big(\,X\,,\,Y\,\big) \overset{\text{indep}}{=} 0$$

$$\mathbf{Cov}\big(\,aX+b\,,\,cY+d\,\big) = ac\mathbf{Cov}\big(\,X\,,\,Y\,\big), \quad \mathbf{Cov}\big(\,\sum_{i=1}^{n} X_i\,,\,\sum_{j=1}^{m} Y_j\,\big) = \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{Cov}\big(\,X_i\,,\,Y_j\,\big)$$

$$\rho(X,Y) = \frac{\mathbf{Cov}\big(\,X\,,\,Y\,\big)}{\sqrt{\mathbf{Var}\big(\,X\,\big)\mathbf{Var}\big(\,Y\,\big)}}$$

Central Limit Theorem:

If $X_1, X_2, \ldots$ are i.i.d. with $\mathbf{E}\big(\,X_i\,\big) = \mu$ and $\mathbf{Var}\big(\,X_i\,\big) = \sigma^2$ and $S_n = X_1 + \cdots + X_n$ then

$$\lim_{n\to\infty} \mathbf{P}\big(\,\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\,\big) = \Phi(x)$$

$$Z \sim \mathcal{N}(0,1): \quad \mathbf{P}\big(\,Z \leq z\,\big) = \Phi(z), \quad \mathbf{P}\big(\,Z \geq z\,\big) = 1-\Phi(z), \quad \Phi(-z) = 1-\Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 1: Values of $\Phi(z)$ (the c.d.f. of an $\mathcal{N}(0,1)$ random variable.)