Limit / large dev. thms. exercises before second midterm

1. Prove that the uniform distribution UNI$[-1,1]$ cannot be expressed as the difference of two i.i.d. random variables. *Hint:* Use the method of characteristic functions!

2. Let X_n be uniformly distributed on the set $\{1,2,\dots,n\}$. Use the method of characteristic functions to show that $X_n/n \Rightarrow \mathrm{UNI}[0,1]$.

3. Use the method of characteristic functions to show that the difference of two independent EXP(1) random variables has the same distribution as XY, where $\mathbb{P}(X=1)=\mathbb{P}(X=-1)=\frac{1}{2}$ and $Y\sim \mathrm{EXP}(1)$ and X and Y are independent.

4. Show by an example that $\phi_{X+Y}(u)=\phi_X(u)\phi_Y(u)$ does not necessarily imply that the random variables X and Y are independent. *Hint:* Think of a famous distribution!

5. Let U, X and Y be independent random variables distributed as follows: $U\sim\mathrm{UNI}[0,1]$, $X,Y\sim\mathrm{EXP}(1)$. Use the method of characteristic functions to prove that
$$Z := U\cdot(X+Y)\sim \mathrm{EXP}(1).$$

6. *The Lévy distribution is stable.* Let X denote a random variable with standard Lévy distribution. On the one hand, we have already learnt that $S_n/n^2 \Rightarrow X$, where $S_n=\eta_1+\cdots+\eta_n$, where $\eta_1,\eta_2,\dots$ are i.i.d. and η_k has the same distribution as the hitting time of level one by a one dimensional simple symmetric random walk starting from the origin. On the other hand, we have learnt that $\mathbb{E}(e^{itX})=e^{-\sqrt{-2it}}$. Denote by LEVY$(a)$ the distribution of aX, where $a\in\mathbb{R}_+$.

   Give two different proofs of the fact that for any $a,b\in\mathbb{R}_+$ we have
$$\mathrm{LEVY}(a)*\mathrm{LEVY}(b)\sim\mathrm{LEVY}((\sqrt{a}+\sqrt{b})^2). \qquad (1)$$
   (The $*$ symbol denotes convolution)

7. Let $X_1,X_2,X_3,\dots$ denote i.i.d. r.v.'s with UNI$[0,1]$ distribution. Use Lindeberg to show that
$$\frac{\sum_{k=1}^{n} kX_k-\frac{n^2}{4}}{\frac{1}{6}n^{\frac{3}{2}}}\Rightarrow N(0,1)$$

8. For any $s\in(1,+\infty)$ let X_s denote an $\mathbb{N}_+$-valued random variable satisfying $\mathbb{P}(X_s=n)=n^{-s}/\zeta(s)$, where $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$. Denote by Y_s the number of distinct primes that divide X_s. Show that
$$\frac{Y_{1+\varepsilon}-\ln(1/\varepsilon)}{\sqrt{\ln(1/\varepsilon)}}\Rightarrow\mathcal{N}(0,1),\qquad \varepsilon\to 0_+ \qquad (2)$$
   Hint: To approximate $\sum_{p\in\mathcal{P}}p^{-s}$, take the log of the Euler formula (see page 128) for the Riemann zeta function $\zeta(s)$.

9. Prove that X_n converges to 0 in probability if and only if $\varphi_n(t)\to 1$ in an open neighbourhood of $t=0$.

10. Let $X_1,X_2,\dots$ be i.i.d. random variables. Assume $\mathbb{P}(X_i\ge 0)=1$, $\mathbb{E}X_i=1$ and $\mathrm{Var}(X_i)=\sigma^2<\infty$. Prove that
$$2\left(\sqrt{S_n}-\sqrt{n}\right)\Rightarrow\mathcal{N}(0,\sigma^2).$$

11. For each $n\in\mathbb{N}$, let $\xi_{n,k}$, $k=1,\dots,n$ denote i.i.d. random variables with BER$(1/n)$ distribution. These random variables form a triangular array. Let $S_n=\xi_{n,1}+\cdots+\xi_{n,n}$. Find the weak limit of
$$\frac{S_n-\mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}},\qquad n\to\infty.$$
    Explain why this is a valuable lesson in the context of Lindeberg's theorem.