Midterm Exam - May 15, 2024, Limit thms. of probab.

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $Y_1, Y_2, \ldots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(Y_i = +1) = \frac{1}{3}, \qquad \mathbb{P}(Y_i = -1) = \frac{1}{3}, \qquad \mathbb{P}(Y_i = 0) = \frac{1}{3}.$$

   Let $Z_0 = 0$ and $Z_n = Y_1 + \cdots + Y_n$. Let $\tau := \min\{n \geq 0 \, : \, Z_n = 1\}$.
   Let $\mathcal{T}_0 := 0$ and let $\mathcal{T}_k := \min\{n > \mathcal{T}_{k-1} \, : \, Z_n = 0\}$.

   (a) Let $z \in \mathbb{C}$ with $|z| \leq 1$. Find $\mathbb{E}[z^\tau]$. *Hint:* You will have to solve a quadratic equation.
   (b) Let $z \in \mathbb{C}$ with $|z| \leq 1$. Find $\mathbb{E}[z^{\mathcal{T}_1}]$.
   (c) Find $\mathbb{E}[z^{\mathcal{T}_k}]$.
   (d) Find the value of $\eta \in \mathbb{R}_+$ such that

$$\mathcal{T}_k / k^\eta \Rightarrow \mathcal{T}$$

   as $k \to \infty$ (where $\mathcal{T}$ is a non-degenerate random variable) and find the characteristic function of $\mathcal{T}$.

2. Let $X_1, X_2, \ldots$ denote independent random variables with distribution

$$\mathbb{P}(X_k = \pm k^2) = \frac{1}{4\sqrt{k}}, \quad \mathbb{P}(X_k = \pm k^3) = \frac{1}{4k^2}, \quad \mathbb{P}(X_k = 0) = 1 - \frac{1}{2\sqrt{k}} - \frac{1}{2k^2}.$$

   Let $S_n = X_1 + \cdots + X_n$.

   (a) Show that Lindeberg's theorem cannot be applied to the above case in order to prove

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \Rightarrow \mathcal{N}(0,1)$$

   because Lindeberg's condition fails.
   (b) Find a, b, α, β such that

$$\frac{S_n - a n^\alpha}{b n^\beta} \Rightarrow \mathcal{N}(0,1). \tag{1}$$

   Hint: use truncation, Borel-Cantelli and Lindeberg (for the truncated random variables).

   Hint: In your calculations you may use without proof that for any $\gamma > -1$ we have

$$1^\gamma + 2^\gamma + \cdots + n^\gamma \approx \frac{n^{\gamma+1}}{\gamma+1}$$

   (in the sense that the ratio of the two sides goes to 1 as $n \to \infty$)