Midterm Exam Solution 2 - May 11, 2022, Limit thms. of probab.

1. Let $\varphi(t) = \mathbb{E}(e^{itX})$ for some random variable X. Which of the following functions are also characteristic functions of random variables?

   (a) (2 points) $\overline{\varphi(3t)}e^{-|t|}$

   (b) (2 points) $1 - \sqrt{1 - \varphi^2(t)}$

   (c) (2 points) $\frac{\mathrm{Re}(\varphi(t))+2\varphi(t)}{3+t^2}$

   (d) (2 points) $\frac{1}{2}\int_{-\infty}^{\infty} \varphi(t/s)e^{-|s|}\,\mathrm{d}s$

   Solution:

   (a) Let Y be independent of X with CAU(1) distribution. Then $\overline{\varphi(3t)}e^{-|t|}$ is the char. fn. of $-3X+Y$.

   (b) Let R denote the first return time of a simple random walk to the origin. We know from the solution of HW7.2(a) that $\mathbb{E}(z^R) = 1 - \sqrt{1-z^2} =: G(z)$. Let $X_1, X_2, \dots$ denote i.i.d. copies of X, independent of R. Then the char. fn. of $X_1 + \dots + X_R$ is $G(\varphi(t)) = 1 - \sqrt{1-\varphi^2(t)}$, similarly to the solution of HW8.2(e).

   (c) $\frac{\mathrm{Re}(\varphi(t))+2\varphi(t)}{3+t^2} = \frac{\mathrm{Re}(\varphi(t))+2\varphi(t)}{3}\frac{3}{3+t^2} = \left(\frac{1}{6}\overline{\varphi(t)} + \frac{5}{6}\varphi(t)\right)\frac{1}{1+(t/\sqrt{3})^2} =: \psi(t)$, thus if X, Y, Z are independent, Y has p.d.f. $\frac{1}{2}e^{-|x|}$ and $\mathbb{P}(Z=-1) = \frac{1}{6}$, $\mathbb{P}(Z=+1) = \frac{5}{6}$, then the char.fn. of $XZ + Y/\sqrt{3}$ is $\psi(t)$ (see HW6.2(a)).

   (d) If Y has p.d.f. $f(x) = \frac{1}{2}e^{-|x|}$ for all $x \in \mathbb{R}$ and Y is independent of X then X/Y has char.fn. $\frac{1}{2}\int_{-\infty}^{\infty}\varphi(t/s)e^{-|s|}\,\mathrm{d}s$, similarly to HW8.2(f).

2. (7 points) Let $X_1, X_2, \dots$ denote i.i.d. random variables with p.d.f. $f(x) = \frac{3}{2}\cdot x^{-4}\mathbb{1}[\,|x| \geq 1\,]$, $x \in \mathbb{R}$. Let $S_n = 1\cdot X_1 + 2\cdot X_2 + \dots n \cdot X_n$. Find a, b, α, β such that

$$\frac{S_n - an^\alpha}{bn^\beta} \Rightarrow \mathcal{N}(0,1) \tag{1}$$

Hint: In your calculation you may use without proof that for any $\gamma > -1$ we have

$$1^\gamma + 2^\gamma + \dots + n^\gamma \approx \frac{n^{\gamma+1}}{\gamma+1} \tag{2}$$

(in the sense that the ratio of the two sides goes to 1 as $n \to \infty$)

Solution: Let X have p.d.f. f. $\mathbb{E}(X) = 0$ by symmetry. Thus $\mathbb{E}(S_n) = 0$ by linearity. Now

$$\mathrm{Var}(X) = \mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x = 2\int_1^\infty \frac{3}{2}x^{-2}\,\mathrm{d}x = 3.$$

Thus $\sigma_n^2 = \mathrm{Var}(S_n) = \sum_{k=1}^n \mathrm{Var}(kX_k) = \sum_{k=1}^n k^2\cdot 3 \approx 3\frac{n^3}{3}$ (using (2)), thus $\sigma_n \approx n^{3/2}$, so if we want to apply Lindeberg's theorem, then $a = 0$, α can be anything, $b = 1$ and $\beta = 3/2$. Let $\xi_{n,k} = k\cdot X_k$, $k = 1, 2, \dots, n$. Then $\tilde{\xi}_{n,k} = \xi_{n,k} - \mathbb{E}(\xi_{n,k}) = k\cdot X_k$. Let us check that Lindeberg's condition holds:

$$\frac{1}{\sigma_n^2}\sum_{k=1}^n \mathbb{E}\Big[|\tilde{\xi}_{n,k}|^2 \cdot \mathbb{1}\left[|\tilde{\xi}_{n,k}| > \varepsilon\sigma_n\right]\Big] = \frac{1}{\sigma_n^2}\sum_{k=1}^n \mathbb{E}\big[(k\cdot X)^2\cdot \mathbb{1}\left[|k\cdot X| > \varepsilon\sigma_n\right]\big] =$$

$$\frac{1}{\sigma_n^2}\sum_{k=1}^n k^2\mathbb{E}\Big[X^2\cdot\mathbb{1}\left[|X| > \frac{\varepsilon\sigma_n}{k}\right]\Big] = \frac{1}{\sigma_n^2}\sum_{k=1}^n k^2 2\int_{\frac{\varepsilon\sigma_n}{k}}^\infty x^2\cdot\frac{3}{2}\cdot x^{-4}\,\mathrm{d}x = \frac{1}{\sigma_n^2}\sum_{k=1}^n k^2 3\frac{k}{\varepsilon\sigma_n} =$$

$$\frac{1}{\varepsilon}\frac{3}{\sigma_n^3}\sum_{k=1}^n k^3 \overset{(2)}{\approx} \frac{1}{\varepsilon}\frac{3}{\sigma_n^3}\frac{n^4}{4} \approx \frac{3}{4\varepsilon}\frac{n^4}{(n^{3/2})^3} = \frac{3}{4\varepsilon}\frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

The conditions of Lindeberg's theorem hold (including the row-wise independence of the triangular array), thus (1) holds by Lindeberg's theorem (and we also used Slutsky when we replaced σ_n by $n^{3/2} = bn^\beta$).