

Midterm Exam 2 - May 11, 2022, Limit thms. of probab.

Family name _____ Given name _____

Signature _____ Neptun Code _____

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $\varphi(t) = \mathbb{E}(e^{itX})$ for some random variable X . Which of the following functions are also characteristic functions of random variables?

- (a) (2 points) $\overline{\varphi(3t)}e^{-|t|}$
- (b) (2 points) $1 - \sqrt{1 - \varphi^2(t)}$
- (c) (2 points) $\frac{\operatorname{Re}(\varphi(t)) + 2\varphi(t)}{3+t^2}$
- (d) (2 points) $\frac{1}{2} \int_{-\infty}^{\infty} \varphi(t/s)e^{-|s|} ds$

Instruction: In case the function is a characteristic function, write down the recipe of the random variable such that the function is the characteristic function of that random variable. In case it is not a characteristic function, explain why.

2. (7 points) Let X_1, X_2, \dots denote i.i.d. random variables with p.d.f. $f(x) = \frac{3}{2} \cdot x^{-4} \mathbf{1}[|x| \geq 1]$, $x \in \mathbb{R}$. Let $S_n = 1 \cdot X_1 + 2 \cdot X_2 + \dots + n \cdot X_n$. Find a, b, α, β such that

$$\frac{S_n - an^\alpha}{bn^\beta} \Rightarrow \mathcal{N}(0, 1) \tag{1}$$

Instruction: In case you use a theorem learnt in class, check the conditions of the theorem.

Hint: In your calculation you may use without proof that for any $\gamma > -1$ we have

$$1^\gamma + 2^\gamma + \dots + n^\gamma \approx \frac{n^{\gamma+1}}{\gamma+1}$$

(in the sense that the ratio of the two sides goes to 1 as $n \rightarrow \infty$)