Midterm Exam - March 31, 2022, Limit thms. of probab. SOLUTIONS

1. Let $Y = X_1 + \cdots + X_{10000}$, where $X_1, \ldots, X_{10000}$ are i.i.d. with distribution

$$\mathbb{P}(X_i = k) = \frac{1}{64}\binom{6}{k-1}, \qquad k = 1, 2, \ldots, 7.$$

   (a) (4 marks) Approximate the probability $\mathbb{P}(Y = 4 \cdot 10^4)$ using known results.

   (b) (4 marks) Approximate $\ln(\mathbb{P}(Y \leq 2 \cdot 10^4))$ using known results.

Solution: Note that if $Z_i := X_i - 1$ then $Z_i \sim \mathrm{BIN}(6, \frac{1}{2})$. If $Z := Z_1 + \cdots + Z_{10000}$ then $Z \sim \mathrm{BIN}(6 \cdot 10^4, \frac{1}{2})$. We have $Y = Z + 10^4$. Note that $\mathbb{E}(Z) = 6 \cdot 10^4 \cdot \frac{1}{2} = 3 \cdot 10^4$.

   (a) $\mathbb{P}(Y = 4 \cdot 10^4) = \mathbb{P}(Z = 3 \cdot 10^4)$. We have $\frac{\sqrt{6 \cdot 10^4}}{2}\mathbb{P}(Z = \frac{6 \cdot 10^4}{2} + \frac{\sqrt{6 \cdot 10^4}}{2} \cdot 0) \approx \frac{1}{\sqrt{2\pi}} e^{-0^2/2}$ by de Moivre's theorem (HW4.3(c)), thus

$$\mathbb{P}(Y = 4 \cdot 10^4) \approx \frac{1}{\sqrt{2\pi}} \frac{2}{100\sqrt{6}} = \frac{1}{100\sqrt{3\pi}}.$$

   (b) We have $\mathbb{P}(Y \leq 2 \cdot 10^4) = \mathbb{P}(Z \leq 10^4)$. We have $\frac{1}{6 \cdot 10^4} \ln\left(\mathbb{P}(\frac{Z}{6 \cdot 10^4} \leq \frac{1}{6})\right) \approx -\min_{x \leq \frac{1}{6}} I(x)$ by Cramér's theorem where $I(x) = x \ln(\frac{x}{p}) + (1-x)\ln(\frac{1-x}{1-p}) = x \ln(2x) + (1-x)\ln(2-2x)$, since $p = \frac{1}{2}$. We have $\min_{x \leq \frac{1}{6}} I(x) = I(\frac{1}{6})$, since $\frac{1}{6} \leq p = \frac{1}{2}$. Thus

$$\ln(\mathbb{P}(Y \leq 2 \cdot 10^4)) \approx -6 \cdot 10^4 \cdot I(\tfrac{1}{6}) = -6 \cdot 10^4 \cdot \left(\tfrac{1}{6}\ln(2\tfrac{1}{6}) + (1 - \tfrac{1}{6})\ln(2 - 2\tfrac{1}{6})\right) = -10^4 \ln(\tfrac{1}{3}) - 5 \cdot 10^4 \ln(\tfrac{5}{3}).$$

2. Let $Y_1, Y_2, \ldots$ denote i.i.d. random variables with density function g. Let us assume that $g : \mathbb{R} \to (0, +\infty)$ is a strictly positive and continuous function (in particular: $\mathbb{P}[Y_i < 0] > 0$ and $\mathbb{P}[Y_i = 0] = 0$). Let

$$M_n = \min\left\{\frac{1}{Y_1}, \ldots, \frac{1}{Y_n}\right\}.$$

Let G denote the c.d.f. of Y_i. Let $\beta \in \mathbb{R}_+$. Let us denote by F_n the c.d.f. of M_n / n^β.

   (a) (4 marks) Calculate $F_n(x)$ for all $x \in \mathbb{R}$. *Hint:* Consider the cases $x > 0$ and $x < 0$ separately.

   (b) (3 marks) Find the value of $\beta \in \mathbb{R}_+$ for which M_n / n^β converges in distribution to a non-degenerate probability distribution as $n \to \infty$ and identify the c.d.f. of the limiting distribution.

Solution:

   (a) Let $y := n^\beta x$. We have $F_n(x) = \mathbb{P}(M_n / n^\beta \leq x) = \mathbb{P}(M_n \leq y)$. If $x < 0$ then $y < 0$ and

$$F_n(x) = \mathbb{P}(M_n \leq y) = 1 - \mathbb{P}(M_n > y) = 1 - \mathbb{P}(\frac{1}{Y_1} > y)^n = 1 - (1 - \mathbb{P}(\frac{1}{Y_1} \leq y))^n =$$
$$1 - (1 - \mathbb{P}(\frac{1}{y} \leq Y_1 < 0))^n = 1 - (1 - (G(0) - G(\tfrac{1}{y})))^n.$$

   If $x > 0$ then $y > 0$ and

$$F_n(x) = \mathbb{P}(M_n \leq y) = 1 - \mathbb{P}(M_n > y) = 1 - \mathbb{P}(\frac{1}{Y_1} > y)^n = 1 - \mathbb{P}(0 \leq Y_1 < \frac{1}{y})^n = 1 - (G(\tfrac{1}{y}) - G(0))^n.$$

   (b) We want to find $\beta > 0$ such that $F_n(x)$ converges to the c.d.f. of a non-degenerate random variable. First note that for any choice of $\beta > 0$ we have $\lim_{n\to\infty} F_n(0) = \lim_{n\to\infty}(1 - (1 - G(0))^n) = 1$, since $G(0) = \int_{-\infty}^0 g(x)\,\mathrm{d}x > 0$. Thus $\lim_{n\to\infty} F_n(x) = 1$ for all $x \geq 0$ for any choice of $\beta > 0$. On the other hand, if $x < 0$ then $G(0) - G(\frac{1}{n^\beta x}) = \int_{n^{-\beta}\frac{1}{x}}^0 g(u)\,\mathrm{d}u = n^{-\beta}\frac{g(0)}{|x|} + o(n^{-\beta})$ and thus

$$\lim_{n\to\infty} F_n(x) = 1 - \lim_{n\to\infty}(1 - (G(0) - G(\tfrac{1}{n^\beta x})))^n = 1 - \exp\left(-\lim_{n\to\infty} n \cdot (G(0) - G(\tfrac{1}{n^\beta x}))\right) \overset{(*)}{=} 1 - \exp\left(\frac{-g(0)}{|x|}\right),$$

   where $(*)$ holds if we choose $\beta = 1$. Thus $M_n / n \Rightarrow M$, where $F(x) = \mathbb{P}(M \leq x)$ satisfies $F(x) = 1 - \exp\left(-\frac{g(0)}{|x|}\right)$ for $x < 0$ and $F(x) = 1$ for $x \geq 0$. It is easy to check that F is indeed a c.d.f.