Midterm Exam - May 3, 2024, Limit thms. of probab.

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. (8 points) Let X be a random variable with distribution

$$\mathbb{P}(X = k) = \frac{1}{e}\frac{1}{(k-1)!}, \qquad k = 1, 2, 3, \ldots$$

Let $X_1, X_2, \ldots$ denote i.i.d. random variables with the same distribution as X. Let us define

$$S_n = X_1 + \cdots + X_n.$$

(a) Find the logarithmic moment generating function $\lambda \mapsto \ln(M(\lambda))$ of X.

(b) Find the tilting parameter $\lambda_3 \in \mathbb{R}$ such that the exponentially tilted random variable $X^{(\lambda_3)}$ has expectation equal to 3.

(c) Find the limit $R_3 = \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left[S_n \geq 3n\right]\right)$.

(d) What is the relation between the values of $\ln(M(\lambda_3))$, λ_3 and R_3 according to Cramér's theorem? Check that this identity between the numbers that you found in (a),(b),(c) above indeed holds.

2. (7 points) Let $Y_1, Y_2, \ldots$ denote independent and identically distributed random variables with optimistic GEO(1/2) distribution. Let

$$M_n = \max\{Y_1, \ldots, Y_n\}.$$

For some $c \in \mathbb{R}_+$ let

$$Z(n) := M_n - c \cdot \ln(n).$$

Let $n_k := 2^k, k = 0, 1, 2, \ldots$.

(a) How to choose the constant c if we want $Z(n_k)$ to converge in distribution as $k \to \infty$? What is the c.d.f. of the limiting distribution?

(b) Does $Z(n)$ converge in distribution as $n \to \infty$ with the above choice of c? Why?