Limit Large Dev. Thms. Exam, June 15, 2022.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. Let X denote a random variable, let $\hat{I}$ denote its logarithmic moment generating function, let I denote the Legendre transform of $\hat{I}$. Let $\underline{x} := \inf\{x : I(x) < +\infty\}$ and $\overline{x} := \sup\{x : I(x) < +\infty\}$. Let $m := \mathbb{E}(X) < a < b \in \mathbb{R}$ and let $x \in (a,b) \cap (\underline{x}, \overline{x})$. Let $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \ldots$ are i.i.d. copies of X.
   (a) Show that $I(x) = x \cdot \lambda^* - \hat{I}(\lambda^*)$, where $\lambda^* := \lambda^*(x) := (\hat{I}')^{-1}(x)$.
   (b) Show that $\mathbb{E}(X^{(\lambda^*)}) = x$.
   (c) Show that $X_1^{(\lambda^*)} + \cdots + X_n^{(\lambda^*)} \sim S_n^{(\lambda^*)}$, where $X_1^{(\lambda^*)}, X_2^{(\lambda^*)}, \ldots$ are i.i.d. copies of $X^{(\lambda^*)}$.
   Instruction: You can assume that X is an integer-valued random variable.
   (d) Show that $\liminf_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}[\,\frac{S_n}{n} \in (a,b)\,]\right) \geq -I(x)$.
   Instruction: Indicate where you used the results of the sub-exercises (a), (b), (c).

2. Let (X_n) denote a one dimensional simple symmetric random walk starting from the origin.
   Let $M_n := \max\{X_0, X_1, \ldots, X_n\}$. Use the reflection principle to show that $M_n/\sqrt{n}$ weakly converges as $n \to \infty$ and identify the limiting distribution.
   Instruction: You can use the central limit theorem in your proof.

3. Let X_n denote an optimistic geometric random variable with success probability $p = 1/n$.
   (a) Show that $X_n/\mathbb{E}(X_n)$ converges in distribution as $n \to \infty$ and identify the limiting distribution. *Instruction:* Show that the cumulative distribution function (c.d.f.) of $X_n/\mathbb{E}(X_n)$ converges pointwise to the c.d.f. of a random variable.
   (b) Show that $X_n/\mathbb{E}(X_n)$ converges in distribution as $n \to \infty$ and identify the limiting distribution. *Instruction:* This time, use the method of characteristic functions to give an alternative proof of the result of sub-exercise (a).

4. Write down everything you know about Holtsmark's problem.