Limit Large Dev. Thms. Exam, June 19, 2018.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. (a) (5 marks) Define the logarithmic moment generating function $\widehat{I}(\lambda)$ of the non-degenerate r.v. X.

   (b) (8 marks) Prove that $\widehat{I}'(\lambda)$ and $\widehat{I}''(\lambda)$ can be expressed using exponentially tilted distributions and show that $\widehat{I}$ is strictly convex.

   (c) (7 marks) Define the large deviation rate function $I(x)$ of the random variable X as the Legendre transform of $\widehat{I}$. This definition expresses $I(x)$ as the solution of an optimization problem. Give an explicit formula for $I(x)$ by solving this optimization problem.

   (d) (5 marks) Use the formula of (c) to calculate $I(m)$, $I'(m)$ and $I''(m)$ where $m = \mathbb{E}(X)$.

2. (a) (7 marks) Define the notion of weak convergence of a sequence of probability distributions on $\mathbb{R}$ (using cumulative distribution functions), i.e., define the notion of $X_n \Rightarrow X$.

   (b) (18 marks) Prove that if $\lim_{n\to\infty} \mathbb{E}(g(X_n)) = \mathbb{E}(g(X))$ for any bounded and continuous $g : \mathbb{R} \to \mathbb{R}$ then $X_n \Rightarrow X$. (The converse implication also holds but you do not have to prove that implication).

3. Let $F(x) = \mathbb{P}(X \le x)$. Let

$$G(x) = \frac{1}{2}\lim_{\varepsilon\to 0} F(x-\varepsilon) + \frac{1}{2}\lim_{\varepsilon\to 0} F(x+\varepsilon) = \frac{1}{2}\left(F(x_-) + F(x)\right), \qquad x \in \mathbb{R}$$

   In particular, if x is a point of continuity of F then $G(x) = F(x)$. Let $\varphi(t) = \mathbb{E}(e^{itX})$. Let Y denote an independent random variable with standard normal distribution. For $\sigma \in \mathbb{R}_+$ let F_σ denote the cumulative distribution function of $X + \sigma Y$. Note that $F_\sigma(x) = \mathbb{E}\left[\Phi\left(\frac{x-X}{\sigma}\right)\right]$, where $\Phi(x) = \mathbb{P}(Y \le x)$.

   (a) (9 marks) Use the dominated convergence theorem to show that $\lim_{\sigma\to 0} F_\sigma(x) = G(x)$.

   (b) (8 marks) For any $a \le b \in \mathbb{R}$ give an integral formula for $F_\sigma(b) - F_\sigma(a)$ in terms of φ.
   Hint: You can use that $f_\sigma(x) = F'_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\Psi_\sigma(t)\,\mathrm{d}t$, where $\Psi_\sigma(t) = \mathbb{E}(e^{it(X+\sigma Y)})$.

   (c) (8 marks) Use (a) and (b) to show that for any $a \le b \in \mathbb{R}$ we have

$$\frac{1}{2}\mathbb{P}(X=a) + \mathbb{P}(a < X < b) + \frac{1}{2}\mathbb{P}(X=b) = \lim_{\sigma\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ibt} - e^{-iat}}{-it}e^{-\sigma^2 t^2/2}\varphi(t)\,\mathrm{d}t.$$

4. (a) (5 marks) Define the notion of a stable probability distribution.

   (b) (7 marks) Use (a) to prove that if X is stable then $aX + b$ is also stable for any $a, b \in \mathbb{R}$.

   (c) (5 marks) Use the method of characteristic functions to show that the Lévy distribution is stable.
   Recall: You can use that the characteristic function of the Lévy distribution is $\varphi(t) = e^{-\sqrt{-2it}}$.

   (d) (8 marks) Recall without proof a limit theorem from the theory of random walks where the Lévy distribution arises as the limit distribution. Use this limit theorem to give an alternative proof of the fact that the Lévy distribution is stable. This alternative proof should not use characteristic functions.