Limit Large Dev. Thms. Exam, May 29 ,2018.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. Let X denote a real-valued random variable.
   (a) (5 marks) Express the logarithmic moment generating function of $aX+b$ in terms of the logarithmic moment generating function of X.
   (b) (5 marks) Let X and Y denote independent random variables. Express the logarithmic moment generating function of $X+Y$ in terms of the logarithmic moment generating functions of X and Y.
   (c) (15 marks) Let $X_1, X_2, \ldots$ denote i.i.d. random variables and let N denote a non-negative integer-valued random variable, which is independent from $X_1, X_2, \ldots$ Let $Y = X_1 + \ldots X_N$. Denote by $\widehat{I}$ the logarithmic moment generating function of X_i and denote by $\widehat{J}$ the logarithmic moment generating function of N. Show that the logarithmic moment generating function of Y is $\widehat{J} \circ \widehat{I}$.

2. Let (X_n) denote simple symmetric random walk on $\mathbb{Z}$. Denote by π_n time the random walk spends on the positive side, i.e., let
$$\pi_n = \#\{\, j \in (0,n] \,:\, X_{j-1} + X_j > 0 \,\}.$$
The global form of Paul Lévy's arsine theorem states that $\frac{\pi_{2n}}{2n} \Rightarrow X$, where
$$\mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x \le 0, \\ \frac{2}{\pi}\arcsin(\sqrt{x}) & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$
The local form of the same theorem states that for any $0 \le x \le 1$ we have
$$\lim_{n\to\infty} n\mathbb{P}(\pi_{2n} = 2\lfloor nx \rfloor) = \frac{1}{\pi}\frac{1}{\sqrt{(1-x)x}}.$$
Prove that the local form of the theorem implies the global form of the theorem. Carefully explain how you used Scheffé's lemma and Slutsky's theorem in your proof.

3. (a) (10 marks) State Lindeberg's theorem.
   (b) (15 marks) Prove that the usual central limit theorem is a special case of Lindeberg's theorem. In other words, show that if the summands are i.i.d. with finite variance then Lindeberg's condition is satisfied. Carefully explain how you used the dominated convergence theorem.

4. Let $X_1, X_2, \ldots$ i.i.d. with p.d.f.
$$f(x) = \frac{\alpha}{2|x|^{\alpha+1}} \mathbb{1}[\,|x| > 1\,], \qquad 0 < \alpha < 2.$$
Let $S_n = X_1 + \cdots + X_n$ and let $Z_n = S_n / n^{1/\alpha}$. Prove that
$$\lim_{n\to\infty} \mathbb{E}(e^{itZ_n}) = e^{-c|t|^\alpha}, \qquad c = \alpha \int_0^\infty \frac{1-\cos(y)}{y^{\alpha+1}}\,dy.$$