Limit Large Dev. Thms. Exam, Sample Exam, 2018 Spring.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. (a) Find the logarithmic moment generating function $\widehat{I}(\lambda)$ of the optimistic GEO(p) distribution (i.e., the number of trials until the first success, if the success probability at one trial is p). Show that

$$\widehat{I}'(\lambda) = 1/(1 - e^{\lambda}(1-p)), \qquad \lambda < -\ln(1-p)$$

   (b) Show that the large deviation rate function of the OPTGEO(p) distribution is

$$I(x) = \begin{cases} (x-1)\ln\left(\frac{x-1}{1-p}\right) - x\ln(x) - \ln(p), & x \geq 1 \\ +\infty, & x < 1. \end{cases}$$

   (c) If $X \sim$ OPTGEO(p), find the distribution of the exponentially tilted random variable $X^{(\lambda)}$.

2. *Birthday paradox.* Let us fix $n \in \mathbb{N}$ and let $X_{n,j}$, $j = 1, 2, \ldots$ be i.i.d. random variables uniformly distributed on $\{1, 2, \ldots, n\}$. Define

$$T_n := \min\{k : \exists j < k,\ X_{n,j} = X_{n,k}\}.$$

   In plain words: T_n is the index j when the first coincidence of the values is observed.

   Prove that for any $x \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{\sqrt{n}} > x\right) = e^{-x^2/2}.$$

3. (a) State the theorem relating the finiteness of the moments of X to the differentiability of the characteristic function φ of X and give an expression for the k'th derivative of φ at zero using the k'th moment of X.

   (b) Prove the theorem in the $k = 1$ case. Explain carefully where and how you used the dominated convergence theorem.

4. (a) State the Erdős-Kac theorem. Use the following notation:

$$Z_n = \sum_{p\in\mathcal{P},\, p\leq n} Y_{n,p}, \qquad T_n = \sum_{p\in\mathcal{P},\, p\leq \alpha_n} Y_{n,p}$$

   (b) Prove why it is OK to prove the theorem for T_n instead of Z_n. You can use without proof that

$$\sum_{p\in\mathcal{P},\, p\leq n} \frac{1}{p} = \ln(\ln(n)) + \mathcal{O}(1), \qquad n \in \mathbb{N}.$$

   (c) Prove that $\sum_{p\in\mathcal{P},\, p\leq n} \frac{1}{p} \geq \ln(\ln(n)) + \mathcal{O}(1)$, $n \in \mathbb{N}$.