Limit/large dev. thms. exercise sheet

February 19, 2024

1. Let us a consider a town with 1000 households, with one garbage can per household. The capacity of one garbage can is 30 kilograms. The average weekly garbage output may differ from household to household. However, we assume that the garbage output of different households are independent. The average weekly garbage output of the town is 10000 kilograms. The capacity of one garbage truck is 1000 kilograms. Garbage is taken away once a week.

   How many garbage trucks does the town need if they want all of the garbage to be taken away in one round with 95% chance? *Hint:* Use Hoeffding's inequality.

2. Let us consider a r.v. X with optimistic geometric distribution: $X \sim \mathrm{GEO}(p)$:

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, 3, \ldots$$

   (a) Find the moment generating function $Z(\lambda)$ of X. In particular: for which values of λ do we have $Z(\lambda) < +\infty$?

   (b) Find the logarithmic moment generating function $\widehat{I}(\lambda)$ of X.

   (c) Show that $\widehat{I}'(\lambda) = \frac{1}{1-e^{\lambda}(1-p)}$.

   (d) Recalling the notation from page 19 of the scanned lecture notes, find $\lambda^*(x)$ if $x > 1$.

   (e) Show that the large deviation rate function $I(x)$ (see page 19 of scanned) is

$$I(x) = \begin{cases} (x-1)\ln\left(\frac{x-1}{1-p}\right) - x\ln(x) - \ln(p) & \text{if } x \geq 1, \\ +\infty & \text{if } x < 1. \end{cases}$$

   (f) Show that the exponentially tilted r.v. $X^{(\mu)} \sim \mathrm{GEO}(p')$ for some $p' = p'(p, \mu)$ and that any $p' \in (0, 1)$ can be obtained by choosing μ from the domain of $Z(\cdot)$ appropriately.