Limit/large dev. thms. HW assignment 9. SOLUTION

1. *The log-normal distribution is not determined by its moments (see page 135 of scanned).*

   (a) Let $X \sim \mathcal{N}(0,1)$ and $Y = e^X$. Prove that

   $$f(x) := \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(Y \le x) = (2\pi)^{-1/2}x^{-1}\exp\{-(\log x)^2/2\}\mathbb{1}_{\{x>0\}}.$$

   This is called the *standard log-normal distribution.*

   (b) Compute all moments $\mathbb{E}(Y^k)$, $k = 1, 2, \dots$.

   (c) Let $a \in [-1,1]$ be a fixed parameter and define $f_a : \mathbb{R} \to \mathbb{R}_+$ as follows

   $$f_a(x) = \begin{cases} 0 & \text{if } x < 0, \\ f(x)\big(1 + a\sin(2\pi\log x)\big) & \text{if } x \ge 0. \end{cases} \tag{1}$$

   Prove that f_a is a probability density function and show that the moments of the corresponding distribution *don't vary with the parameter* $a \in [-1,1]$. Thus, these different distributions have the same sequence of moments.
   Hint: Show $\int_0^\infty x^k f(x)\sin(2\pi\log x)\,\mathrm{d}x = 0,\ k \in \mathbb{N}$ by substituting $x = \exp(s+k)$.

Solution:

(a) Let $y \in \mathbb{R}_+$. Let $F(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln(y)) = \Phi(\ln(y))$. Then

$$f(y) = \frac{\mathrm{d}}{\mathrm{d}y}F(y) = \varphi(\ln(y))\frac{1}{y} = \frac{1}{\sqrt{2\pi}}e^{-(\ln(y))^2/2}\frac{1}{y}.$$

Note that we will not use this density function in the rest of this exercise – we will calculate everything using the standard normal distribution.

(b) $\mathbb{E}(Y^k) = \mathbb{E}(e^{kX}) = M(k) = e^{k^2/2}$, where $M(\lambda)$ is the moment generating function of X, calculated in class, see page 14 of the scanned lecture notes.

(c) f_a is non-negative, since $1 + a\sin(2\pi\ln(x)$ is non-negative for any $a \in [-1,1]$ and any $x > 0$. Recall that we denote by $\mathrm{Im}(z)$ the imaginary part of the complex number z.

$$\int_0^\infty x^k f(x)\sin(2\pi\log x)\,\mathrm{d}x = \mathbb{E}(Y^k\sin(2\pi\log Y)) = \mathbb{E}(e^{kX}\sin(2\pi X)) = \mathbb{E}(e^{kX}\mathrm{Im}(e^{2\pi iX})) =$$
$$\mathbb{E}(\mathrm{Im}(e^{(2\pi i+k)X})) = \mathrm{Im}\left(\mathbb{E}(e^{(2\pi i+k)X})\right) = \mathrm{Im}(M(2\pi i+k)) = \mathrm{Im}(e^{(2\pi i+k)^2/2}) =$$
$$\mathrm{Im}(e^{-(2\pi)^2/2}e^{2\pi ik}e^{k^2/2}) \stackrel{(*)}{=} \mathrm{Im}(e^{-(2\pi)^2/2}e^{k^2/2}) = 0,$$

where in $(*)$ we used that $e^{2\pi ik} = 1$ if $k \in \mathbb{N}$. Thus we have

$$\int_0^\infty x^k f_a(x)\,\mathrm{d}x \stackrel{(1)}{=} \int_0^\infty x^k f(x)\,\mathrm{d}x + a\int_0^\infty x^k f(x)\sin(2\pi\log x)\,\mathrm{d}x = \mathbb{E}(Y^k) + 0.$$

In particular, $\int_0^\infty f_a(x)\,\mathrm{d}x = 1$, thus f_a is indeed a probability density function.

Remark: Note that $\lim_{k\to\infty}\left(\mathbb{E}(Y^k)/k!\right)^{1/k} = +\infty$ follows from $\ln(\mathbb{E}(Y^k)) = k^2/2$ and $\ln(k!) \approx k\ln(k)$ (see page 4 of scanned), thus the result of this exercise does not contradict the lemma stated and proved on page 135-136 of the scanned.

2. Let X be a random variable and denote by $\varphi(t) := \mathbb{E}(e^{itX})$ $(t \in \mathbb{R})$ its characteristic function. Let us assume that $-X \sim X$, i.e., we assume that the distribution of X is symmetric.

   (a) Show that if $\limsup_{t \to 0} \big(1 - \varphi(t)\big)/t^2 < +\infty$ then $\mathbb{E}(X^2) < +\infty$.
   Hint: For any $u \in \mathbb{R}_+$ let $f_u(t) := \frac{u^3 t^2 e^{-ut}}{2}$. Calculate $\int_0^\infty \frac{1-\cos(tx)}{t^2} f_u(t)\,\mathrm{d}t$ and use the monotone convergence theorem to show that $\lim_{u\to\infty} \int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t = \frac{1}{2}\mathbb{E}(X^2)$.

   (b) Show that if $\mathbb{E}(X^2) < +\infty$ then $\lim_{t\to 0} \big(1-\varphi(t)\big)/t^2 = \frac{1}{2}\mathbb{E}(X^2)$. *Hint:* Dominated convergence.

   (c) Show that $\varphi(t) = e^{-c|t|^\alpha}$ cannot be the characteristic function of a probability distribution if $\alpha > 2$.

Solution:

(a) First note that $\int_0^\infty f_u(t)\,\mathrm{d}t = 1$, because $f_u(\cdot)$ is the p.d.f. of the sum of three i.i.d. $\mathrm{EXP}(u)$ random variables (see HW3.3). Now let us calculate

$$\int_0^\infty \frac{1-\cos(tx)}{t^2} f_u(t)\,\mathrm{d}t = \frac{u^3}{2}\int_0^\infty (1-\cos(tx))e^{-ut}\,\mathrm{d}t = \frac{u^3}{2}\int_0^\infty (1-\mathrm{Re}(e^{itx}))e^{-ut}\,\mathrm{d}t =$$
$$\frac{u^3}{2}\mathrm{Re}\left(\int_0^\infty e^{-ut} - e^{(ix-u)t}\,\mathrm{d}t\right) = \frac{u^3}{2}\mathrm{Re}\left(\frac{1}{u} + \frac{1}{ix-u}\right) = \frac{u^3}{2}\mathrm{Re}\left(\frac{1}{u} + \frac{-ix-u}{x^2+u^2}\right) =$$
$$\frac{u^3}{2}\left(\frac{1}{u} + \frac{-u}{x^2+u^2}\right) = \frac{1}{2}\left(u^2 - \frac{u^4}{x^2+u^2}\right) = \frac{1}{2}\left(\frac{x^2u^2 + u^4 - u^4}{x^2+u^2}\right) = \frac{1}{2}\frac{x^2u^2}{x^2+u^2}.$$

Since $-X \sim X$, we have $\varphi(t) = \mathbb{E}(\cos(tX))$, see page 86 of the lecture notes. Thus

$$\int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t = \int_0^\infty \frac{1-\mathbb{E}(\cos(tX))}{t^2} f_u(t)\,\mathrm{d}t = \int_0^\infty \mathbb{E}\left(\frac{1-\cos(tX)}{t^2} f_u(t)\right)\,\mathrm{d}t \overset{(*)}{=}$$
$$\mathbb{E}\left(\int_0^\infty \frac{1-\cos(tX)}{t^2} f_u(t)\,\mathrm{d}t\right) = \frac{1}{2}\mathbb{E}\left(\frac{X^2u^2}{X^2+u^2}\right) = \frac{1}{2}\mathbb{E}\left(\frac{X^2}{X^2/u^2+1}\right),$$

where in $(*)$ we note that Fubini is applicable because the integrand is non-negative (see page 114). Now note that $\frac{X^2}{X^2/u^2+1} \geq 0$ and $\frac{X^2}{X^2/u^2+1}$ is an increasing function of u, thus by the monotone convergence theorem (see page 36) we have

$$\lim_{u\to\infty}\int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t = \lim_{u\to\infty}\frac{1}{2}\mathbb{E}\left(\frac{X^2}{X^2/u^2+1}\right) = \frac{1}{2}\mathbb{E}\left(\lim_{u\to\infty}\frac{X^2}{X^2/u^2+1}\right) = \frac{1}{2}\mathbb{E}(X^2).$$

Also note that if $\limsup_{t\to 0}\big(1-\varphi(t)\big)/t^2 < +\infty$ then actually $\sup_{t\in\mathbb{R}}\big(1-\varphi(t)\big)/t^2 = M < +\infty$, therefore for any $u \in \mathbb{R}_+$ we have

$$\int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t \leq \int_0^\infty M f_u(t)\,\mathrm{d}t = M\int_0^\infty f_u(t)\,\mathrm{d}t = M,$$

thus $\mathbb{E}(X^2) \leq 2M < +\infty$.

(b) $\big(1-\varphi(t)\big)/t^2 = \mathbb{E}\left((1-\cos(tX))/t^2\right)$ and $0 \leq 1-\cos(y) \leq \frac{1}{2}y^2$, thus $0 \leq (1-\cos(tX))/t^2 \leq \frac{1}{2}X^2$. Now if $\mathbb{E}(X^2) < +\infty$ then by dominated convergence (see page 37) we have

$$\lim_{t\to 0}\frac{1-\varphi(t)}{t^2} = \lim_{t\to 0}\mathbb{E}\left(\frac{1-\cos(tX)}{t^2}\right) = \mathbb{E}\left(\lim_{t\to 0}\frac{1-\cos(tX)}{t^2}\right) \overset{(**)}{=} \mathbb{E}(\frac{1}{2}X^2),$$

where $(**)$ follows by applying L'Hospital's rule twice.

(c) If $\varphi(t) = e^{-c|t|^\alpha}$, where $\alpha > 2$, and if we indirectly assume that φ is the characteristic function of some random variable X then $\lim_{t\to 0}\frac{1-\varphi(t)}{t^2} = 0$ implies (using (a) and (b)) that $\mathbb{E}(X^2) = 0$, thus $\mathbb{P}(X=0) = 1$, thus $\varphi(t) \equiv 1$, a contradiction. See page 147. See also HW7.1(b).

Remark: Let us assume $-X \sim X$. Putting together (a) and (b) we see that $\limsup_{t\to 0}\big(1-\varphi(t)\big)/t^2 < +\infty$ if and only if $\mathbb{E}(X^2) < +\infty$. However let us note that $\limsup_{t\to 0}\big(1-\varphi(t)\big)/|t| < +\infty$ is not equivalent to $\mathbb{E}(|X|) < +\infty$, as we now explain. First note that $\mathbb{E}(|X|) < +\infty$ does imply that φ is differentiable (see page 90) and thus $\lim_{t\to 0}\big(1-\varphi(t)\big)/t = -\varphi'(0) = -i\mathbb{E}(X) = 0$. But if $X \sim \mathrm{CAU}(1)$ then $\varphi(t) = e^{-|t|}$ and thus $\limsup_{t\to 0}\big(1-\varphi(t)\big)/|t| = 1$, however $\mathbb{E}(|X|) = +\infty$, see page 105. Note that if $-X \sim X$, then $\mathbb{E}(|X|) = \frac{2}{\pi}\int_0^\infty \frac{1-\varphi(t)}{t^2}\mathrm{d}t$ (even if one side is infinite), see page 113-114.

3. Let $X_1, X_2, \ldots$ be independent random variables with the following distributions

$$\mathbb{P}(X_m = \pm m) = \frac{1}{2m^\beta}, \quad \mathbb{P}(X_m = 0) = \frac{m^\beta - 1}{m^\beta},$$

and denote $S_n = X_1 + \cdots + X_n$. Prove the following statements.

(a) If $\beta > 1$, then there exists a random variable S_∞, so that $S_n \to S_\infty$, almost surely.
Hint: Use Borel-Catelli.

(b) If $\beta < 1$, then

$$\frac{S_n}{cn^{(3-\beta)/2}} \Rightarrow \mathcal{N}(0,1), \tag{2}$$

with some appropriately chosen constant $c \in (0, \infty)$. Find c. *Hint:* Use Lindeberg.

(c) If $\beta = 1$, then $S_n/n \Rightarrow \xi$, where ξ is a random variable with characteristic function

$$\mathbb{E}\left(e^{it\xi}\right) = \exp\left(-\int_0^1 \frac{1-\cos(ts)}{s} ds\right). \tag{3}$$

Solution:

(a) If $\beta > 1$, then $\sum_{m=1}^\infty \mathbb{P}(X_m \neq 0) = \sum_{m=1}^\infty m^{-\beta} < \infty$, and due to the Borel-Cantelli Lemma $\nu := \max\{m : X_m \neq 0\} < \infty$, almost surely. Therefore $\lim_{n\to\infty} \sum_{m=1}^n X_m = \sum_{m=1}^\nu X_m =: S_\infty$ exists almost surely.

(b) If $\beta \in (0,1)$ we apply Lindeberg. First note that $\mathbb{E}(X_m) = 0$ and $\mathrm{Var}(X_m) = \mathbb{E}(X_m^2) = m^{2-\beta}$.

$$\sigma_n^2 := \mathrm{Var}(S_n) = \sum_{m=1}^n m^{2-\beta} = \int_0^n x^{2-\beta}\,\mathrm{d}x + \mathcal{O}(n^{2-\beta}) = \frac{n^{3-\beta}}{3-\beta} + \mathcal{O}(n^{2-\beta}).$$

We check Lindeberg's condition. Let $\varepsilon > 0$ be fixed. For n sufficiently large, $n^2 < \varepsilon\sigma_n^2$ (since $\beta < 1$), and for $m \le n$, we have $\mathbb{P}(X_m^2 \le n^2) = 1$, thus $\mathbb{P}(X_m^2 < \varepsilon\sigma_n^2) = 1$. Therefore

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{m=1}^n \mathbb{E}[X_m^2 \mathbb{1}[X_m^2 > \varepsilon\sigma_n^2]] = 0.$$

From this (and Slutsky) the result (2) follows, with the value of the constant $c = (3-\beta)^{-1/2}$.

(c) We first compute the characteristic functions. We have $\varphi_m(t) := \mathbb{E}(e^{itX_m}) = 1 - \frac{1-\cos(mt)}{m}$ and

$$\mathbb{E}(e^{itS_n/n}) = \prod_{m=1}^n \varphi_m(t/n) = \prod_{m=1}^n \left(1 - \frac{1-\cos(\frac{m}{n}t)}{m}\right) = \prod_{m=1}^n \left(1 - \frac{1}{n}\frac{1-\cos(\frac{m}{n}t)}{\frac{m}{n}}\right) \tag{4}$$

We will use HW5.2 to show $\lim_{n\to\infty} \mathbb{E}(e^{itS_n/n}) = \mathbb{E}\left(e^{it\xi}\right)$ (this will also show that the r.h.s. of (3) is indeed the characteristic function of a random variable by the theorem stated on page 108).
Using the notation of HW5.2 let us fix $t \in [0, +\infty)$ and define $y_{n,m} = -\frac{1}{n}\frac{1-\cos(\frac{m}{n}t)}{\frac{m}{n}}$.
Observe that $y_{n,m} \le 0$ and $\sup_{s>0} \frac{1-\cos(s)}{s} < \infty$ from which $\lim_{n\to\infty} \max_{1\le m\le n} |y_{n,m}| = 0$ follows.
Also note that $\lim_{n\to\infty} \sum_{m=1}^n y_{n,m} = -\int_0^1 \frac{1-\cos(ts)}{s}\,\mathrm{d}s$ by the definition of Riemannian integration.
This also shows that $\sup_n \sum_{m=1}^n |y_{n,m}| < \infty$.
Thus all of the conditions of HW5.2 are satisfied, thus $\lim_{n\to\infty} \mathbb{E}(e^{itS_n/n}) = \mathbb{E}\left(e^{it\xi}\right)$ follows.

Remark: We did not have time to better understand the limiting distribution ξ that appears in (3) because the semester is over. Let me just say that the distribution of ξ is a so-called *infinitely divisible distribution* and ξ also appears in a slightly modified version of the one-dimensional Holtsmark problem.