Limit/large dev. thms. HW assignment 9. Due Wednesday, May 8.

Note: Each of the 3 questions is worth 15 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. *The log-normal distribution is not determined by its moments (see page 135 of scanned).*

   (a) Let $X \sim \mathcal{N}(0,1)$ and $Y = e^X$. Prove that

   $$f(x) := \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(Y \le x) = (2\pi)^{-1/2} x^{-1} \exp\{-(\log x)^2/2\} \mathbb{1}_{\{x>0\}}.$$

   This is called the *standard log-normal distribution.*

   (b) Compute all moments $\mathbb{E}(Y^k)$, $k = 1, 2, \dots$.

   (c) Let $a \in [-1,1]$ be a fixed parameter and define $f_a : \mathbb{R} \to \mathbb{R}_+$ as follows

   $$f_a(x) = \begin{cases} 0 & \text{if } x < 0, \\ f(x)\big(1 + a\sin(2\pi \log x)\big) & \text{if } x \ge 0. \end{cases}$$

   Prove that f_a is a probability density function and show that the moments of the corresponding distribution *don't vary with the parameter* $a \in [-1,1]$. Thus, these different distributions have the same sequence of moments.
   Hint: Show $\int_0^\infty x^k f(x) \sin(2\pi \log x)\,\mathrm{d}x = 0,\ k \in \mathbb{N}$ by substituting $x = \exp(s+k)$.

2. Let X be a random variable and denote by $\varphi(t) := \mathbb{E}(e^{itX})$ $(t \in \mathbb{R})$ its characteristic function. Let us assume that $-X \sim X$, i.e., we assume that the distribution of X is symmetric.

   (a) Show that if $\limsup_{t\to 0} \big(1 - \varphi(t)\big)/t^2 < +\infty$ then $\mathbb{E}(X^2) < +\infty$.
   Hint: For any $u \in \mathbb{R}_+$ let $f_u(t) := \frac{u^3 t^2 e^{-ut}}{2}$. Calculate $\int_0^\infty \frac{1-\cos(tx)}{t^2} f_u(t)\,\mathrm{d}t$ and use the monotone convergence theorem to show that $\lim_{u\to\infty} \int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t = \frac{1}{2}\mathbb{E}(X^2)$.

   (b) Show that if $\mathbb{E}(X^2) < +\infty$ then $\lim_{t\to 0} \big(1 - \varphi(t)\big)/t^2 = \frac{1}{2}\mathbb{E}(X^2)$. *Hint:* Dominated convergence.

   (c) Show that $\varphi(t) = e^{-c|t|^\alpha}$ cannot be the characteristic function of a probability distribution if $\alpha > 2$.

3. Let $X_1, X_2, \dots$ be independent random variables with the following distributions

   $$\mathbb{P}(X_m = \pm m) = \frac{1}{2m^\beta}, \quad \mathbb{P}(X_m = 0) = \frac{m^\beta - 1}{m^\beta},$$

   and denote $S_n = X_1 + \dots + X_n$. Prove the following statements.

   (a) If $\beta > 1$, then there exists a random variable S_∞, so that $S_n \to S_\infty$, almost surely.
   Hint: Use Borel-Catelli.

   (b) If $\beta < 1$, then

   $$\frac{S_n}{cn^{(3-\beta)/2}} \Rightarrow \mathcal{N}(0,1),$$

   with some appropriately chosen constant $c \in (0,\infty)$. Find c. *Hint:* Use Lindeberg.

   (c) If $\beta = 1$, then $S_n/n \Rightarrow \xi$, where ξ is a random variable with characteristic function

   $$\mathbb{E}\left(e^{it\xi}\right) = \exp\left(-\int_0^1 \frac{1-\cos(ts)}{s} ds\right).$$