Limit/large dev. thms. HW assignment 8. SOLUTION

1. Let $F(x) = \mathbb{P}(X \leq x)$. Recall from page 39 of the scanned lecture notes that F is right-continuous. Let

$$G(x) = \frac{1}{2}\lim_{\varepsilon\to 0} F(x-\varepsilon) + \frac{1}{2}\lim_{\varepsilon\to 0} F(x+\varepsilon) = \frac{1}{2}\left(F(x_-) + F(x)\right), \qquad x \in \mathbb{R}$$

In particular, if x is a point of continuity of F then $G(x) = F(x)$. Let $\varphi(t) = \mathbb{E}(e^{itX})$. Let Y denote an independent random variable with standard normal distribution. For $\sigma \in \mathbb{R}_+$ let F_σ denote the cumulative distribution function of $X + \sigma Y$ (see page 102 of the scanned lecture notes).

(a) Use the dominated convergence theorem to show that $\lim_{\sigma\to 0} F_\sigma(x) = G(x)$.

Hint: Use one of the formulas for F_σ from page 102 of the scanned lecture notes.

(b) For any $a \leq b \in \mathbb{R}$ give an integral formula for $F_\sigma(b) - F_\sigma(a)$ in terms of φ.

Hint: Use Fubini and the lemma from page 103 which gives a formula for $f_\sigma = F'_\sigma$ in terms of φ.

(c) Use (a) and (b) to show that for any $a \leq b \in \mathbb{R}$ we have

$$\frac{1}{2}\mathbb{P}(X=a) + \mathbb{P}(a < X < b) + \frac{1}{2}\mathbb{P}(X=b) = \lim_{\sigma\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ibt} - e^{-iat}}{-it}e^{-\sigma^2 t^2/2}\varphi(t)\,\mathrm{d}t.$$

(d) Assume further that the distribution of X is absolutely continuous and denote by f the p.d.f. of X. Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Recall that we denote by f_σ the p.d.f. of $X + \sigma Y$. Write down a formula for f_σ using convolution (see page 20 of the scanned lecture notes) and show that $\lim_{\sigma\to 0} f_\sigma(x) = f(x)$ for any $x \in \mathbb{R}$ (see page 104).

Remark: f is not necessarily bounded! Also note that the solution of part (d) of this exercise has nothing to do with the solution of the results of parts (a),(b) and (c).

Solution:

(a) We know from page 102 of the scanned lecture notes that $F_\sigma(x) = \mathbb{E}(\Phi(\frac{x-X}{\sigma}))$, where $\Phi(x)$ denotes the c.d.f. of $\mathcal{N}(0,1)$. Let us fix $x \in \mathbb{R}$. Note that $0 \leq \Phi(\frac{x-X}{\sigma}) \leq 1$ and

$$\lim_{\sigma\to 0}\Phi\left(\frac{x-X}{\sigma}\right) = \begin{cases} 0 & \text{if } X > x, \\ \frac{1}{2} & \text{if } X = x, \\ 1 & \text{if } X < x. \end{cases}$$

In other words: $\lim_{\sigma\to 0}\Phi(\frac{x-X}{\sigma}) = \frac{1}{2}\mathbb{1}[X = x] + \mathbb{1}[X < x] = \frac{1}{2}\left(\mathbb{1}[X < x] + \mathbb{1}[X \leq x]\right)$. Therefore, by dominated convergence we get

$$\lim_{\sigma\to 0} F_\sigma(x) = \lim_{\sigma\to 0}\mathbb{E}\left[\Phi\left(\frac{x-X}{\sigma}\right)\right] = \mathbb{E}\left[\lim_{\sigma\to 0}\Phi\left(\frac{x-X}{\sigma}\right)\right] =$$
$$\mathbb{E}\left[\frac{1}{2}\left(\mathbb{1}[X < x] + \mathbb{1}[X \leq x]\right)\right] = \frac{1}{2}\left(\mathbb{P}[X < x] + \mathbb{P}[X \leq x]\right) = \frac{1}{2}\left(F(x_-) + F(x)\right) = G(x)$$

(b) In the equation marked by $(*)$ below, we use Fubini, which is applicable, since $|e^{-itx}e^{-t^2\sigma^2/2}\varphi(t)| \leq e^{-\sigma^2 t^2/2}$, and $\int_a^b\int_{-\infty}^{+\infty} e^{-t^2\sigma^2/2}\,\mathrm{d}t\,\mathrm{d}x < +\infty$:

$$F_\sigma(b) - F_\sigma(a) = \int_a^b f_\sigma(x)\,\mathrm{d}x = \int_a^b \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}e^{-t^2\sigma^2/2}\varphi(t)\,\mathrm{d}t\,\mathrm{d}x \overset{(*)}{=}$$
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t^2\sigma^2/2}\varphi(t)\int_a^b e^{-itx}\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2 t^2/2}\varphi(t)\frac{e^{-ibt} - e^{-iat}}{-it}\,\mathrm{d}t.$$

(c) $\lim_{\sigma\to 0}\left(F_\sigma(b) - F_\sigma(a)\right) = G(b) - G(a) = \frac{1}{2}\mathbb{P}(X = a) + \mathbb{P}(a < X < b) + \frac{1}{2}\mathbb{P}(X = b)$

(d) We denote by φ_σ the p.d.f. of $\mathcal{N}(0,\sigma^2)$, thus $\varphi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$. Let us fix $x \in \mathbb{R}$. It is enough to show that

$$\limsup_{\sigma\to 0}|f_\sigma(x) - f(x)| \leq \epsilon \tag{1}$$

for any $\varepsilon > 0$, so let us fix $\varepsilon > 0$ and let $\delta > 0$ be so small that $|f(y) - f(x)| \leq \varepsilon$ if $|y - x| \leq \delta$. Then $f_\sigma(x) = \int_{-\infty}^{\infty} f(y)\varphi_\sigma(x-y)\,\mathrm{d}y = A_\sigma + B_\sigma$, where

$$A_\sigma = \int_{x-\delta}^{x+\delta} f(y)\varphi_\sigma(x-y)\,\mathrm{d}y, \qquad B_\sigma = \int_{-\infty}^{\infty} \mathbb{1}\left[|y-x| > \delta\right] f(y)\varphi_\sigma(x-y)\,\mathrm{d}y. \tag{2}$$

In order to prove (1), it is enough to show $\lim_{\sigma\to 0} B_\sigma = 0$ and $\limsup_{\sigma\to 0} |A_\sigma - f(x)| \leq \varepsilon$.

$$\lim_{\sigma\to 0} B_\sigma \leq \lim_{\sigma\to 0} \left(\sup_{|z|\geq\delta} \varphi_\sigma(z) \right) \cdot \int_{-\infty}^{\infty} \mathbb{1}\left[|y-x| > \delta\right] f(y)\,\mathrm{d}y \leq 0 \cdot 1 = 0. \tag{3}$$

It remains to show $\limsup_{\sigma\to 0} |A_\sigma - f(x)| \leq \varepsilon$.

$$A_\sigma - f(x) = \int_{x-\delta}^{x+\delta} f(y)\varphi_\sigma(x-y)\,\mathrm{d}y - \int_{-\infty}^{\infty} f(x)\varphi_\sigma(x-y)\,\mathrm{d}y = \\ \int_{x-\delta}^{x+\delta} (f(y) - f(x))\varphi_\sigma(x-y)\,\mathrm{d}y - f(x)\int_{-\infty}^{\infty} \mathbb{1}\left[|y-x| > \delta\right] \varphi_\sigma(x-y)\,\mathrm{d}y. \tag{4}$$

The last term on the r.h.s. of (4) goes to zero as $\sigma \to 0$, because $\int_{-\infty}^{\infty} \mathbb{1}\left[|y-x| > \delta\right] \varphi_\sigma(x-y)\,\mathrm{d}y = \mathbb{P}(|(x - \sigma Y) - x| > \delta)$, where $Y \sim \mathcal{N}(0,1)$, so it remains to bound

$$\left| \int_{x-\delta}^{x+\delta} (f(y) - f(x))\varphi_\sigma(x-y)\,\mathrm{d}y \right| \leq \int_{x-\delta}^{x+\delta} |f(y) - f(x)|\,\varphi_\sigma(x-y)\,\mathrm{d}y \leq \int_{x-\delta}^{x+\delta} \varepsilon\varphi_\sigma(x-y)\,\mathrm{d}y \leq \varepsilon. \tag{5}$$

This completes the proof of $\limsup_{\sigma\to 0} |A_\sigma - f(x)| \leq \varepsilon$.

2. Let $\varphi(t) = \mathbb{E}(e^{itX})$. Show that the following functions are also characteristic functions:

$$\text{(a) } \overline{\varphi}(t), \quad \text{(b) } \varphi^2(t), \quad \text{(c) } |\varphi(t)|^2, \quad \text{(d) } \mathrm{Re}(\varphi(t)), \quad \text{(e) } \frac{1}{2-\varphi(t)}, \quad \text{(f) } \int_0^\infty \varphi(st)e^{-s}\,\mathrm{d}s$$

Hint: You don't have to use Bochner, each of these formulas have a probabilistic meaning.

Solution:

(a) $\overline{\varphi}(t)$ is the characteristic function of $-X$. Indeed: $\mathbb{E}(e^{it(-X)}) = \mathbb{E}(e^{i(-t)X}) = \overline{\varphi}(t)$, see page 87 of the scanned lecture notes.

(b) $\varphi^2(t)$ is the characteristic function of $X_1 + X_2$, where X_1 and X_2 are i.i.d. copies of X. Indeed: $\mathbb{E}(e^{it(X_1+X_2)}) = \mathbb{E}(e^{itX_1}e^{itX_2}) = \mathbb{E}(e^{itX_1})\mathbb{E}(e^{itX_2}) = \varphi(t)\varphi(t)$.

(c) $|\varphi(t)|^2$ is the characteristic function of $X_1 - X_2$, where where X_1 and X_2 are i.i.d. copies of X. Indeed: $\mathbb{E}(e^{it(X_1-X_2)}) = \mathbb{E}(e^{itX_1}e^{it(-X_2)}) = \mathbb{E}(e^{itX_1})\mathbb{E}(e^{it(-X_2)}) = \varphi(t)\overline{\varphi}(t) = |\varphi(t)|^2$.

(d) $\mathrm{Re}(\varphi(t))$ is the characteristic function of XY, where Y is independent from X and $\mathbb{P}(Y=1) = \mathbb{P}(Y=-1) = \frac{1}{2}$. Indeed: $\mathbb{E}(e^{itXY}) = \frac{1}{2}\mathbb{E}(e^{itX}) + \frac{1}{2}\mathbb{E}(e^{it(-X)}) = \frac{1}{2}\left(\varphi(t) + \overline{\varphi}(t)\right) = \mathrm{Re}(\varphi(t))$.

(e) $\frac{1}{2-\varphi(t)}$ is the characteristic function of $X_1 + X_2 + \cdots + X_N$, where $X_1, X_2, \ldots$ are i.i.d. copies of X and N is an independent random variable with pessimistic $\mathrm{GEO}(\frac{1}{2})$ distribution. First note that the generating function of N is

$$G(z) = \mathbb{E}(z^N) = \sum_{k=0}^\infty z^k \mathbb{P}(N=k) = \sum_{k=0}^\infty z^k 2^{-(k+1)} = \frac{1}{2}\sum_{k=0}^\infty \left(\frac{z}{2}\right)^k = \frac{1}{2}\frac{1}{1-z/2} = \frac{1}{2-z}.$$

Then we can calculate

$$\mathbb{E}\left(e^{it(X_1+X_2+\cdots+X_N)}\right) = \sum_{k=0}^\infty \mathbb{E}\left(e^{it(X_1+X_2+\cdots+X_k)}\right)\mathbb{P}(N=k) =$$

$$\sum_{k=0}^\infty (\varphi(t))^k\,\mathbb{P}(N=k) = G(\varphi(t)) = \frac{1}{2-\varphi(t)}$$

(f) $\int_0^\infty \varphi(st)e^{-s}\,\mathrm{d}s$ is the characteristic function of XY, where $Y \sim \mathrm{EXP}(1)$ is independent from X. Indeed, the density function of Y is $f(y) = e^{-y}\mathbb{1}[y \geq 0]$ and

$$\mathbb{E}(e^{itXY}) = \int_0^\infty \mathbb{E}(e^{itXY}|Y=y)f(y)\,\mathrm{d}y = \int_0^\infty \mathbb{E}(e^{itXy})f(y)\,\mathrm{d}y = \int_0^\infty \varphi(yt)e^{-y}\,\mathrm{d}y$$

3. (a) Give a probabilistic meaning to the following trigonometric identity by interpreting both sides as a characteristic function:

$$\frac{\sin(t)}{t} = \cos(t/2)\frac{\sin(t/2)}{t/2}.$$

(b) By iterating the identity in (a) prove the following trigonometric identity:

$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right).$$

(c) Provide probabilistic interpretation (i.e. probabilistic proof) of the identity in (b).

Solution:

(a) If $Y \sim \text{UNI}[-\frac{1}{2}, \frac{1}{2}]$ then $\varphi_Y(t) = \frac{\sin(t/2)}{t/2}$ (see page 89 of the scanned lecture notes).
If $\mathbb{P}(X = \frac{1}{2}) = \mathbb{P}(X = -\frac{1}{2}) = \frac{1}{2}$, then $\varphi_X(t) = \frac{1}{2}e^{it/2} + \frac{1}{2}e^{-it/2} = \cos(t/2)$.
Now $Y - \frac{1}{2} \sim \text{UNI}[-1, 0]$ and $Y + \frac{1}{2} \sim \text{UNI}[0, 1]$, whence if X and Y are independent, then $X + Y \sim \text{UNI}[-1, 1]$, therefore $\varphi_{X+Y}(t) = \sin(t)/t$. On the other hand, we have $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$, thus

$$\frac{\sin(t)}{t} = \varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = \cos(t/2)\frac{\sin(t/2)}{t/2}$$

(b) Noting that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, we obtain

$$\frac{\sin(t)}{t} = \cos(t/2)\frac{\sin(t/2)}{t/2} = \cos(t/2)\cos(t/4)\frac{\sin(t/4)}{t/4} = \cdots = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right).$$

(c) We know that if $Z \sim \text{UNI}[0, 1]$ and if $\eta_1, \eta_2, \ldots$ are the digits in the binary expansion of Z, i.e.

$$Z = \sum_{n=1}^{\infty} \eta_n 2^{-n}$$

then $\eta_1, \eta_2, \ldots$ are i.i.d. with $\text{BER}(\frac{1}{2})$ distribution, i.e., $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 0) = \frac{1}{2}$.
Thus

$$2Z - 1 = 2\sum_{n=1}^{\infty} \eta_n 2^{-n} - \sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{\infty} 2^{-n}\left(2\eta_n - 1\right).$$

Now the characteristic function of $2\eta_n - 1$ is $\cos(t)$, thus

$$\frac{\sin(t)}{t} = \mathbb{E}(e^{it(2Z-1)}) = \mathbb{E}(e^{it(\sum_{n=1}^{\infty} 2^{-n}(2\eta_n - 1))}) = \prod_{n=1}^{\infty} \mathbb{E}(e^{it2^{-n}(2\eta_n - 1)}) = \prod_{n=1}^{\infty} \cos\left(\frac{t}{2^n}\right).$$