Limit/large dev. thms. HW assignment 8. Due Wednesday, April 24

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. Let $F(x) = \mathbb{P}(X \le x)$. Recall from page 39 of the scanned lecture notes that F is right-continuous. Let

$$G(x) = \frac{1}{2}\lim_{\varepsilon\to 0} F(x-\varepsilon) + \frac{1}{2}\lim_{\varepsilon\to 0} F(x+\varepsilon) = \frac{1}{2}\left(F(x_-) + F(x)\right), \qquad x \in \mathbb{R}$$

In particular, if x is a point of continuity of F then $G(x) = F(x)$. Let $\varphi(t) = \mathbb{E}(e^{itX})$. Let Y denote an independent random variable with standard normal distribution. For $\sigma \in \mathbb{R}_+$ let F_σ denote the cumulative distribution function of $X + \sigma Y$ (see page 102 of the scanned lecture notes).

   (a) Use the dominated convergence theorem to show that $\lim_{\sigma\to 0} F_\sigma(x) = G(x)$.
   Hint: Use one of the formulas for F_σ from page 102 of the scanned lecture notes.

   (b) For any $a \le b \in \mathbb{R}$ give an integral formula for $F_\sigma(b) - F_\sigma(a)$ in terms of φ.
   Hint: Use Fubini and the lemma from page 103 which gives a formula for $f_\sigma = F'_\sigma$ in terms of φ.

   (c) Use (a) and (b) to show that for any $a \le b \in \mathbb{R}$ we have

$$\frac{1}{2}\mathbb{P}(X=a) + \mathbb{P}(a < X < b) + \frac{1}{2}\mathbb{P}(X=b) = \lim_{\sigma\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ibt} - e^{-iat}}{-it} e^{-\sigma^2 t^2/2}\varphi(t)\,\mathrm{d}t.$$

   (d) Assume further that the distribution of X is absolutely continuous and denote by f the p.d.f. of X. Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Recall that we denote by f_σ the p.d.f. of $X + \sigma Y$. Write down a formula for f_σ using convolution (see page 20 of the scanned lecture notes) and show that $\lim_{\sigma\to 0} f_\sigma(x) = f(x)$ for any $x \in \mathbb{R}$ (see page 104).
   Remark: f is not necessarily bounded! Also note that the solution of part (d) of this exercise has nothing to do with the solution of the results of parts (a),(b) and (c).

2. Let $\varphi(t) = \mathbb{E}(e^{itX})$. Show that the following functions are also characteristic functions:

$$\text{(a) } \overline{\varphi}(t), \quad \text{(b) } \varphi^2(t), \quad \text{(c) } |\varphi(t)|^2, \quad \text{(d) } \mathrm{Re}(\varphi(t)), \quad \text{(e) } \frac{1}{2-\varphi(t)}, \quad \text{(f) } \int_0^\infty \varphi(st)e^{-s}\,\mathrm{d}s$$

Hint: You don't have to use Bochner, each of these formulas have a probabilistic meaning.

3. (a) Give a probabilistic meaning to the following trigonometric identity by interpreting both sides as a characteristic function:

$$\frac{\sin(t)}{t} = \cos(t/2)\frac{\sin(t/2)}{t/2}.$$

   (b) By iterating the identity in (a) prove the following trigonometric identity:

$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty}\cos\left(\frac{t}{2^k}\right).$$

   (c) Provide probabilistic interpretation (i.e. probabilistic proof) of the identity in (b).