Limit/large dev. thms. HW assignment 7. Due Wednesday, April 17.

Note: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. (15 marks) Decide about the following functions $\mathbb{R} \to \mathbb{C}$ whether they are characteristic functions of probability distributions or not.

$$(a)\ \frac{1}{1+t^2}, \quad (b)\ \exp(-t^4), \quad (c)\ \sin(t), \quad (d)\ \cos(t), \quad (e)\ \frac{1+\cos t}{2}, \quad (f)\ \frac{\sin(t)}{t} \quad (g)\ 2\frac{1-\cos(t)}{t^2}$$

Hint: You will NOT need to use Bochner's theorem (page 88), especially that we did not even learn it. To show that a function is not a characteristic function, you need to use the properties that we have learnt in class. To show that it is a characteristic function of a random variable X, you have to find the distribution of X. It is a wise idea to read the official solution of HW6.2 before you start solving this exercise.

2. (10 marks) In this exercise $\sqrt{z}$ denotes the complex analytic function which is defined for all complex numbers except for the negative real numbers in the following way: if $\mathrm{Im}(z) \geq 0$ then $\arg(\sqrt{z}) = \frac{1}{2}\arg(z)$ and $|\sqrt{z}| = \sqrt{|z|}$, moreover we extend the function to the half-plane $\mathrm{Im}(z) \leq 0$ by the identity $\sqrt{\overline{z}} = \overline{\sqrt{z}}$. Then of course $z \geq 0$ implies $\sqrt{z} \geq 0$, so this complex function $\sqrt{\cdot} : \mathbb{C} \setminus \{-\mathbb{R}_+\} \to \mathbb{C}$ is an analytic extension of the usual square root function $\sqrt{\cdot} : \mathbb{R}_+ \to \mathbb{R}_+$

We consider simple symmetric random walk on $\mathbb{Z}$. Recall that we denote by T_k the time it takes to reach level k (see page 59 of the scanned lecture notes). Recall that we denote by R_k the time of the k'th return to the origin (see page 65). We have found the generating function of T_1 in class (page 99).

   (a) Find the generating function of R_k. *Hint:* See page 65 of lecture notes.

   (b) Recall from page 65 that R_k/k^2 weakly converges to the Lévy distribution (defined on page 61) as $k \to \infty$. Use this to show that the characteristic function of the Lévy distribution is $\exp(-\sqrt{-2it})$. *Hint:*

$$\lim_{n\to\infty}(1-a_n)^n = e^{-z} \quad \text{if} \quad \lim_{n\to\infty} n \cdot a_n = z$$

3. (15 marks) *Coupon collector's problem.* Suppose that there is an urn with n different coupons in it. You start to draw coupons from the urn with replacement. In each round you pick each coupon with equal probability. Denote by V_n the number of coupons that you need to draw until you can say that you have touched all of the coupons at least once. The goal of this exercise is to prove the limit theorem

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{V_n - n\ln(n)}{n} \leq x\right) = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}. \tag{1}$$

   (a) Let $\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n}$ denote independent random variables with distribution

$$\mathbb{P}(\xi_{n,k} = m) = \frac{k}{n}\left(\frac{n-k}{n}\right)^{m-1}, \quad m = 1, 2, \dots$$

   Show that V_n has the same distribution as $\xi_{n,1} + \xi_{n,2} + \dots + \xi_{n,n}$ by giving a probabilistic meaning to the random variables $\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n}$ in the context of coupon collection.
   Hint: This is very similar to the lemma proved on page 96-97 of the scanned lecture notes.

   (b) Find $\lim_{n\to\infty} \mathbb{E}\left(\frac{V_n - n\ln(n)}{n}\right)$ and $\lim_{n\to\infty} \mathrm{Var}\left(\frac{V_n - n\ln(n)}{n}\right)$.

   (c) Show that for any fixed $k \in \mathbb{N}$, we have $\xi_{n,k}/n \Rightarrow \mathrm{EXP}(k)$ as $n \to \infty$ using the method of characteristic functions.

   (d) Show that $\frac{V_n - \mathbb{E}(V_n)}{n} \Rightarrow Z$, where $Z + \gamma$ has standard Gumbel distribution and γ is the Euler constant.
   Hint: Use the method of characteristic functions and the results proved in class (see page 95-98).

   (e) Conclude the proof of the result stated in equation (1).