Limit/large dev. thms. HW assignment 6. Due April 10.

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. Let $f : [0,1] \to \mathbb{R}$ be continuous. Find the value of the following limits

$$\text{a) } \lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_1\, dx_2 \cdots dx_n$$

$$\text{b) } \lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2 \cdots dx_n$$

   Hint: There is a probabilistic interpretation to this exercise. Use the equivalent characterization of weak convergence given on page 81 of the scanned lecture notes, similarly to the exercise solved on page 84 of the scanned lecture notes.

2. *Characteristic function of (a) symmetrized exponential density function and (b) „rooftop" density function.*

   Compute the characteristic function of the absolute continuous probability distributions with the following probability density functions on $\mathbb{R}$:

$$\text{(a)} \quad \frac{a}{2} \exp^{-a|x|}, \qquad \text{(b)} \quad \max\{\, a(1 - a\,|x|),\, 0\,\},$$

   where a is a positive constant.

   Hint: This exercise can be viewed a painful integration exercise, but if you are clever enough, you can solve it without calculating integrals (but you will have to use results and tricks from class, see page 85-89 of the scanned lecture notes)!

3. *Explicit error bounds for the relation of characteristic functions and moments*

   (a) Show that if $f : \mathbb{R} \to \mathbb{C}$ is three times continuously differentiable then

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \frac{1}{2}\int_0^x f'''(u)(x-u)^2\, \mathrm{d}u.$$

   (b) Show that

$$e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} = \frac{i^3}{2}\int_0^x (x-u)^2 e^{iu}\, \mathrm{d}u$$

   (c) Show that

$$e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} = -\int_0^x (x-u)(e^{iu} - 1)\, \mathrm{d}u$$

   (d) Show that for all $x \in \mathbb{R}$ we have

$$\left| e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} \right| \le \min\left\{ \frac{|x|^3}{6},\, |x|^2 \right\}$$

   (e) Show that if $\varphi(t)$ is the characteristic function of the random variable X and $\mathbb{E}(X^2) < +\infty$ then

$$\left| \varphi(t) - 1 - it\mathbb{E}(X) + \frac{t^2 \mathbb{E}(X^2)}{2} \right| \le \mathbb{E}\left(\min\left\{ \frac{|t|^3}{6}|X|^3,\, t^2|X|^2 \right\} \right)$$